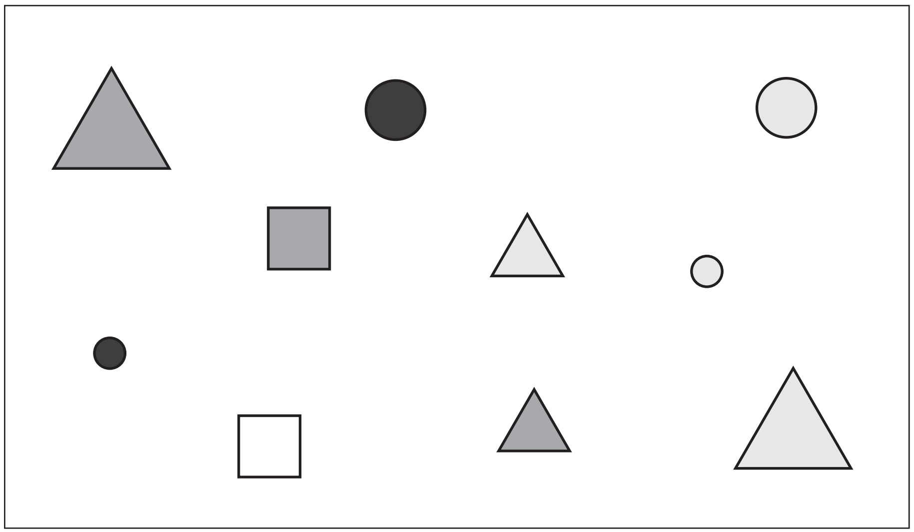
Decomposition

Example World



Relation

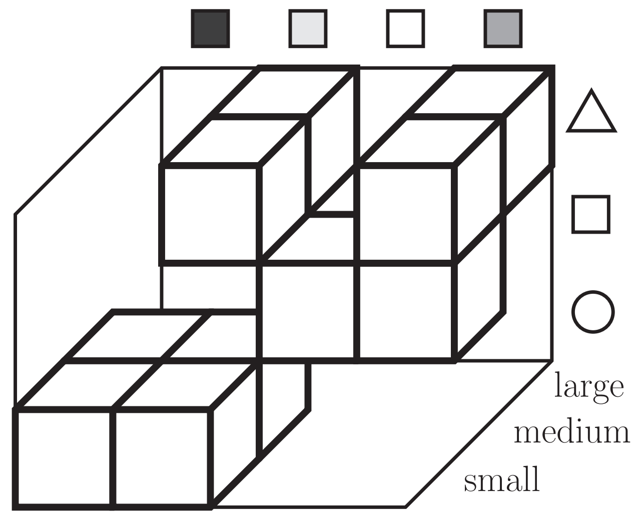
color	shape	size
■	○	small
■	○	medium
▢	○	small
▢	○	medium
▢	△	medium
▢	△	large
□	□	medium
▩	□	medium
▩	△	medium
▩	△	large

- 10 simple geometric objects
- 3 attributes

Relation

color	shape	size
■	○	small
■	○	medium
░	○	small
░	○	medium
░	△	medium
░	△	large
□	□	medium
▒	□	medium
▒	△	medium
▒	△	large

Geometric Representation



Object Representation

Universe of Discourse: Ω

$\omega \in \Omega$ represents a single abstract object.

A subset $E \subseteq \Omega$ is called an **event**.

For every event we use the function R to determine whether E is possible or not.

$$R: \; 2^\Omega \to \{0, 1\}$$

We claim the following properties of R:

1. $R(\emptyset) = 0$
2. $\forall E_1, E_2 \subseteq \Omega: \; R(E_1 \cup E_2) \; = \; \max\{R(E_1), R(E_2)\}$

For example:

$$R(E) \; = \; \begin{cases} 0 & \text{if} \;\; E = \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Object Representation

Attributes or Properties of these objects are introduced by functions:
(later referred to as **random variables**)

$$A: \ \Omega \rightarrow \mathrm{dom}(A)$$

where $\mathrm{dom}(A)$ is the domain (i. e., set of all possible values) of A.

A set of attibutes $U = \{A_1, \ldots, A_n\}$ is called an **attribute schema**.

The **preimage** of an attribute defines an **event**:

$$\forall a \in \mathrm{dom}(A): \ A^{-1}(a) = \{\omega \in \Omega \mid A(\omega) = a\} \subseteq \Omega$$

Abbreviation: $A^{-1}(a) \quad = \quad \{\omega \in \Omega \mid A(\omega) = a\} \quad = \quad \{A = a\}$

We will index the function R to stress on which events it is defined.
R_{AB} will be short for $R_{\{A,B\}}$.

$$R_{AB}: \bigcup_{a \in \mathrm{dom}(A)} \bigcup_{b \in \mathrm{dom}(B)} \Big\{\{A = a, B = b\}\Big\} \rightarrow \{0, 1\}$$

Formal Representation

A = color	B = shape	C = size
$a_1 = \blacksquare$	$b_1 = \bigcirc$	c_1 = small
$a_1 = \blacksquare$	$b_1 = \bigcirc$	c_2 = medium
$a_2 = \square$	$b_1 = \bigcirc$	c_1 = small
$a_2 = \square$	$b_1 = \bigcirc$	c_2 = medium
$a_2 = \square$	$b_3 = \triangle$	c_2 = medium
$a_2 = \square$	$b_3 = \triangle$	c_3 = large
$a_3 = \square$	$b_2 = \square$	c_2 = medium
$a_4 = \square$	$b_2 = \square$	c_2 = medium
$a_4 = \square$	$b_3 = \triangle$	c_2 = medium
$a_4 = \square$	$b_3 = \triangle$	c_3 = large

$$\begin{aligned}
&R_{ABC}(A = a, B = b, C = c)\\
&\quad = R_{ABC}(\{A = a, B = b, C = c\})\\
&\quad = R_{ABC}(\{\omega \in \Omega \mid A(\omega) = a \wedge\\
&\qquad\qquad\qquad B(\omega) = b \wedge\\
&\qquad\qquad\qquad C(\omega) = c)\}\\
&\quad = \begin{cases} 0 & \text{if there is no tuple } (a, b, c)\\ 1 & \text{else} \end{cases}
\end{aligned}$$

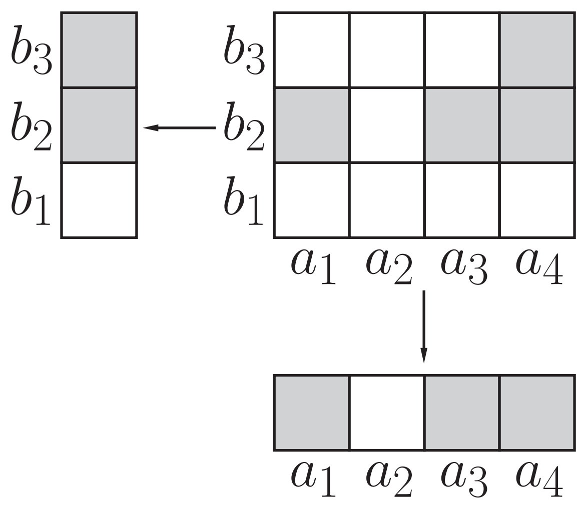
R serves as an indicator function.

Projection / Marginalization

Let R_{AB} be a relation over two attributes A and B. The projection (or marginalization) from schema $\{A, B\}$ to schema $\{A\}$ is defined as:

$$\forall a \in \mathrm{dom}(A): \; R_A(A = a) \; = \; \max_{\forall b \in \mathrm{dom}(B)} \{R_{AB}(A = a, B = b)\}$$

This principle is easily generalized to sets of attributes.

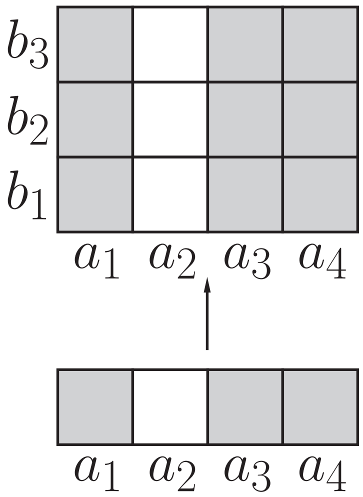


Cylindrical Extention

Let R_A be a relation over an attribute A. The cylindrical extention R_{AB} from $\{A\}$ to $\{A, B\}$ is defined as:

$$\forall a \in \mathrm{dom}(A) : \forall b \in \mathrm{dom}(B) : \; R_{AB}(A = a, B = b) \; = \; R_A(A = a)$$

This principle is easily generalized to sets of attributes.



Object Representation

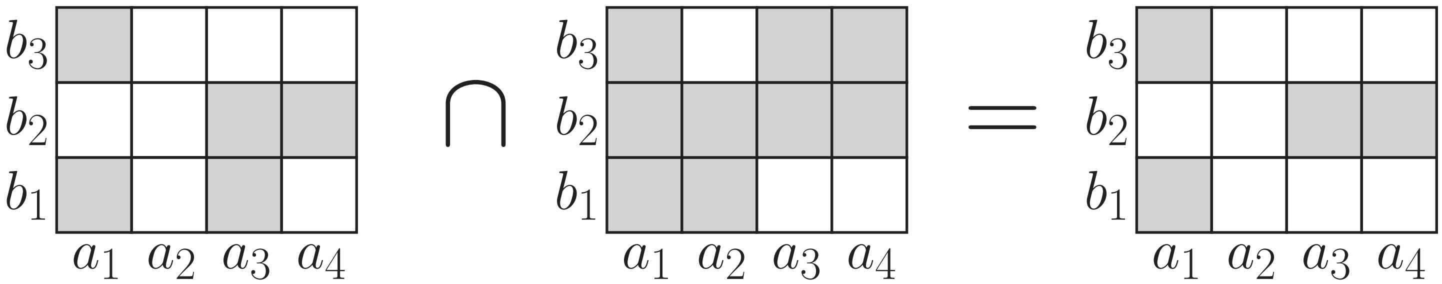
Intersection

Let $R_{AB}^{(1)}$ and $R_{AB}^{(2)}$ be two relations with attribute schema $\{A, B\}$. The intersection R_{AB} of both is defined in the natural way:

$$\forall a \in \mathrm{dom}(A) : \forall b \in \mathrm{dom}(B) :$$

$$R_{AB}(A = a, B = b) \;=\; \min\{R_{AB}^{(1)}(A = a, B = b), R_{AB}^{(2)}(A = a, B = b)\}$$

This principle is easily generalized to sets of attributes.

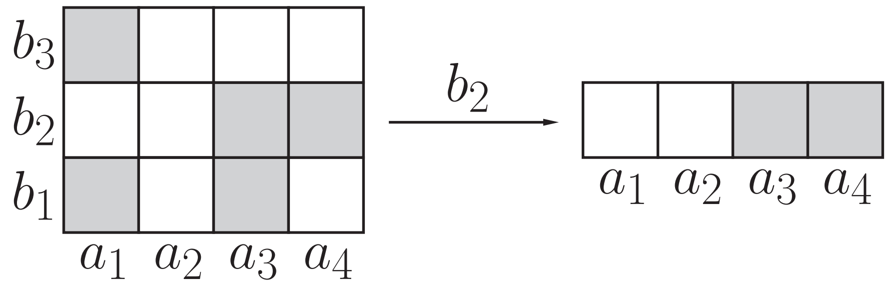


Conditional Relation

Let R_{AB} be a relation over the attribute schema $\{A, B\}$. The conditional relation of A given B is defined as follows:

$$\forall a \in \mathrm{dom}(A) : \forall b \in \mathrm{dom}(B) : R_A(A = a \mid B = b) \;=\; R_{AB}(A = a, B = b)$$

This principle is easily generalized to sets of attributes.

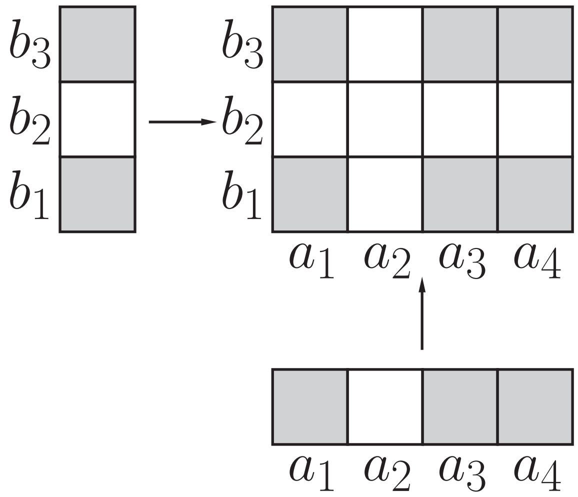


(Unconditional) Independence

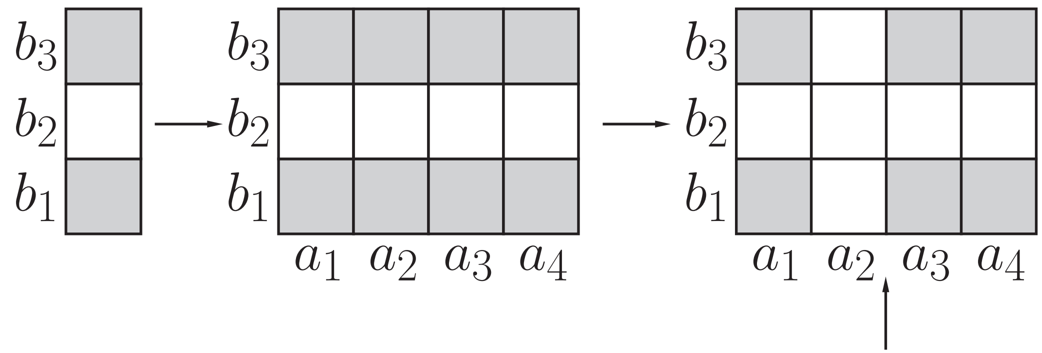
Let R_{AB} be a relation over the attribute schema $\{A, B\}$. We call A and B relationally independent (w. r. t. R_{AB}) if the following condition holds:

$$\forall a \in \mathrm{dom}(A) : \forall b \in \mathrm{dom}(B) : R_{AB}(A = a, B = b) \;=\; \min\{R_A(A = a), R_B(B = b)\}$$

This principle is easily generalized to sets of attributes.

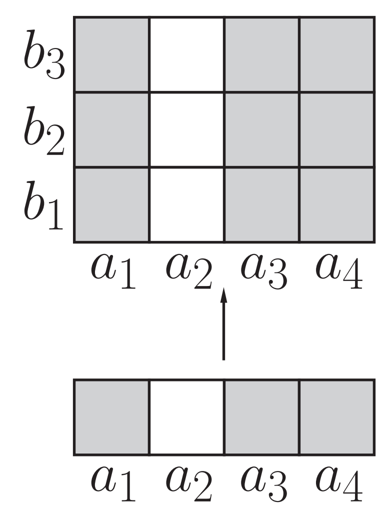


(Unconditional) Independence



Intuition: Fixing one (possible) value of A does not restrict the (possible) values of B and vice versa.

Conditioning on any possible value of B always results in the same relation R_A.



Alternative independence expression:

$$\forall b \in \mathrm{dom}(B): \;\; R_B(B = b) = 1: \\ R_A(A = a \mid B = b) \;=\; R_A(A = a)$$

Decomposition

Obviously, the original two-dimensional relation can be reconstructed from the two one-dimensional ones, if we have (unconditional) independence.

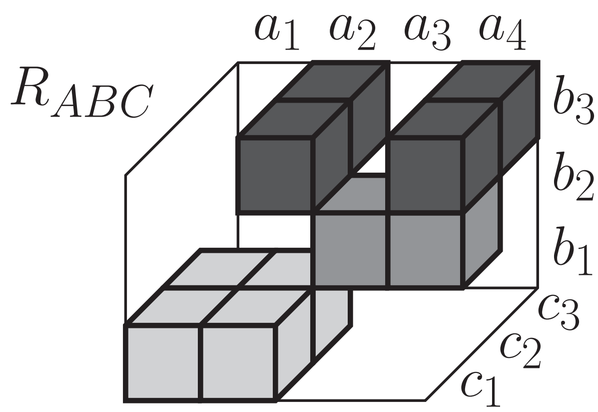
The definition for (unconditional) independence already told us how to do so:

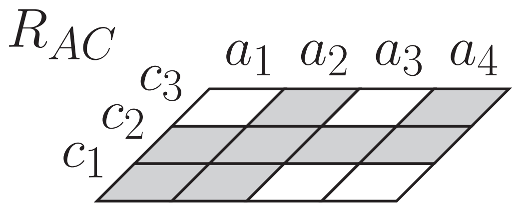
$$R_{AB}(A = a, B = b) \;=\; \min\{R_A(A = a), R_B(B = b)\}$$

Storing R_A and R_B is sufficient to represent the information of R_{AB}.

Question: The (unconditional) independence is a rather strong restriction. Are there other types of independence that allow for a decomposition as well?

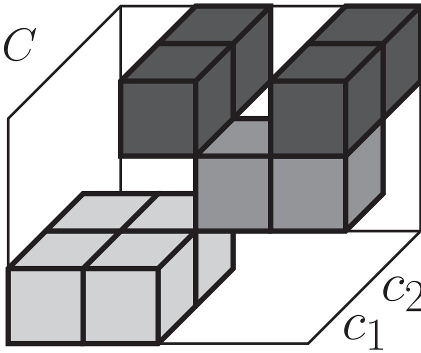
Conditional Relational Independence

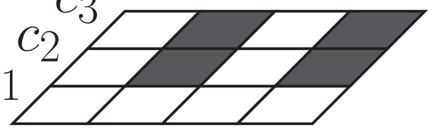




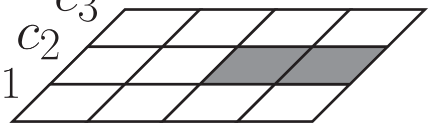
Clearly, A and C are unconditionally dependent, i. e. the relation R_{AC} cannot be reconstructed from R_A and R_C.

Conditional Relational Independence

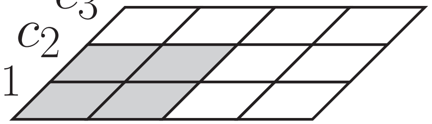




$R_{AC}(\ \cdot, \cdot \mid B = b_3)$



$R_{AC}(\ \cdot, \cdot \mid B = b_2)$



$R_{AC}(\ \cdot, \cdot \mid B = b_1)$

However, given all possible values of B, all respective conditional relations R_{AC} show the independence of A and C.

$$R_{AC}(a, c \mid b) \ = \ \min\{R_A(a \mid b), R_C(c \mid b)\}$$

With the definition of a conditional relation, the decomposition description for R_{ABC} reads:

$$R_{ABC}(a, b, c) \ = \ \min\{R_{AB}(a, b), R_{BC}(b, c)\}$$

Conditional Relational Independence

Again, we reconstruct the initial relation from the cylindrical extentions of the two relations formed by the attributes A, B and B, C.

It is possible since A and C are (relationally) independent given B.