Probability Foundations

Reminder: Probability Theory

Goal: Make statements and/or predictions about results of physical processes.

Even processes that seem to be simple at first sight may reveal considerable difficulties when trying to predict.

Describing real-world physical processes always calls for a simplifying mathematical model.

Although everybody will have some intuitive notion about probability, we have to formally define the underlying mathematical structure.

Randomness or chance enters as the incapability of precisely modelling a process or the inability of measuring the initial conditions.

- *Example*: Predicting the trajectory of a billard ball over more than 9 banks requires more detailed measurement of the initial conditions (ball location, applied momentum etc.) than physically possible according to Heisenberg's uncertainty principle.

Formal Approach on the Model Side

We conduct an experiment that has a set Ω of possible outcomes.
E. g.:

- Rolling a die ($\Omega = \{1, 2, 3, 4, 5, 6\}$)
- Arrivals of phone calls ($\Omega = \mathbb{N}_0$)
- Bread roll weights ($\Omega = \mathbb{R}_+$)

Such an outcome is called an **elementary event**.

All possible elementary events are called the **frame of discernment** Ω (or sometimes **universe of discourse**).

The set representation stresses the following facts:

- All possible outcomes are covered by the elements of Ω. (**collectively exhaustive**).
- Every possible outcome is represented by exactly one element of Ω. (**mutual disjoint**).

Events

Often, we are interested in *higher-level* events
(e. g. casting an odd number, arrival of at least 5 phone calls or purchasing a bread roll heavier than 80 grams)

Any subset $A \subseteq \Omega$ is called an **event** which **occurs**, if the outcome $\omega_0 \in \Omega$ of the random experiment lies in A:

$$\text{Event } A \subseteq \Omega \ \text{ occurs} \quad \Leftrightarrow \quad \bigvee_{\omega \in A} (\omega = \omega_0) \ = \ \mathsf{true} \quad \Leftrightarrow \quad \omega_0 \in A$$

Since events are sets, we can define for two events A and B:

- $A \cup B$ occurs if A or B occurs; $A \cap B$ occurs if A and B occurs.
- $\overline{A}$ occurs if A does not occur (i. e., if $\Omega \backslash A$ occurs).
- A and B are *mutually exclusive*, iff $A \cap B = \emptyset$.

Event Algebra

A family of sets $\mathcal{E} = \{E_1, \ldots, E_n\}$ is called an **event algebra**, if the following conditions hold:

- The **certain event** Ω lies in $\mathcal{E}$.
- If $E \in \mathcal{E}$, then $\overline{E} = \Omega \backslash E \in \mathcal{E}$.
- If E_1 and E_2 lie in $\mathcal{E}$, then $E_1 \cup E_2 \in \mathcal{E}$ and $E_1 \cap E_2 \in \mathcal{E}$.

If Ω is uncountable, we require the additional property:

For a series of events $E_i \in \mathcal{E}, i \in \mathbb{N},$ the events $\bigcup\limits_{i=1}^{\infty} E_i$ and $\bigcap\limits_{i=1}^{\infty} E_i$ are also in $\mathcal{E}$.

$\mathcal{E}$ is then called a **σ-algebra**.

Side remarks:

Smallest event algebra: $\mathcal{E} = \{\emptyset, \Omega\}$

Largest event algebra (for finite or countable Ω): $\mathcal{E} = 2^{\Omega} = \{A \subseteq \Omega \mid \mathsf{true}\}$

Probability Function

Given an event algebra $\mathcal{E}$, we would like to assign every event $E \in \mathcal{E}$ its probability with a **probability function** $P : \mathcal{E} \to [0,1]$.

We require P to satisfy the so-called **Kolmogorov Axioms**:

- $\forall E \in \mathcal{E} : \ 0 \ \leq \ P(E) \ \leq \ 1$
- $P(\Omega) \ = \ 1$
- For pairwise disjoint events $E_1, E_2, \ldots \in \mathcal{E}$ holds:

$$P\Big(\bigcup_{i=1}^{\infty} E_i\Big) \ = \ \sum_{i=1}^{\infty} P(E_i)$$

From these axioms one can conclude the following (incomplete) list of properties:

- $\forall E \in \mathcal{E} : \ P(\overline{E}) \ = \ 1 - P(E)$
- $P(\emptyset) \ = \ 0$
- If $E_1, E_2 \in \mathcal{E}$ are mutually exclusive, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Elementary Probabilities and Densities

Question 1: How to calculate P?

Question 2: Are there "default" event algebras?

Idea for question 1: We have to find a way of distributing (thus the notion *distribution*) the unit mass of probability over all elements $\omega \in \Omega$.

- If Ω is finite or countable a **probability mass function** p is used:

$$p: \ \Omega \to [0,1] \quad \text{and} \quad \sum_{\omega \in \Omega} p(\omega) = 1$$

- If Ω is uncountable (i. e., continuous) a **probability density function** f is used:

$$f: \ \Omega \to \mathbb{R} \quad \text{and} \quad \int_{\Omega} f(\omega)\, \mathrm{d}\omega = 1$$

“Default” Event Algebras

Idea for question 2 (“default” event algebras) we have to distinguish again between the cardinalities of Ω:

- Ω finite or countable: $\mathcal{E} = 2^\Omega$
- Ω uncountable, e. g. $\Omega = \mathbb{R}$: $\mathcal{E} = \mathcal{B}(\mathbb{R})$

$\mathcal{B}(\mathbb{R})$ is the **Borel Algebra**, i. e., the smallest σ-algebra that contains all closed intervals $[a, b] \subset \mathbb{R}$ with $a < b$.

$\mathcal{B}(\mathbb{R})$ also contains all open intervals and single-item sets.

It is sufficient to note here, that all intervals are contained

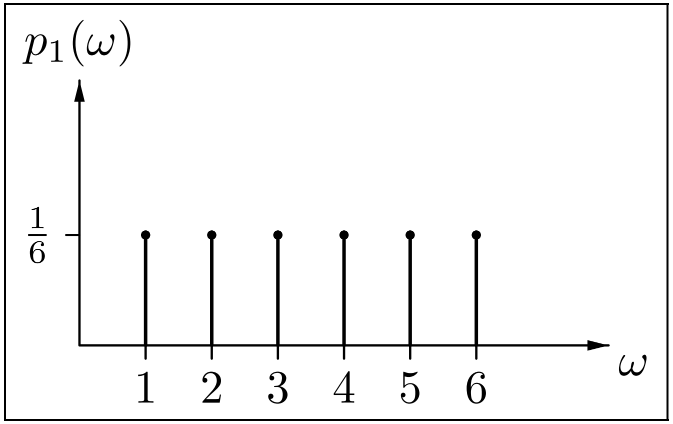
$$\{[a,b]\,,]a,b]\,,]a,b[\,, [a,b[\, \subset \mathbb{R} \mid a < b\} \subset \mathcal{B}(\mathbb{R})$$

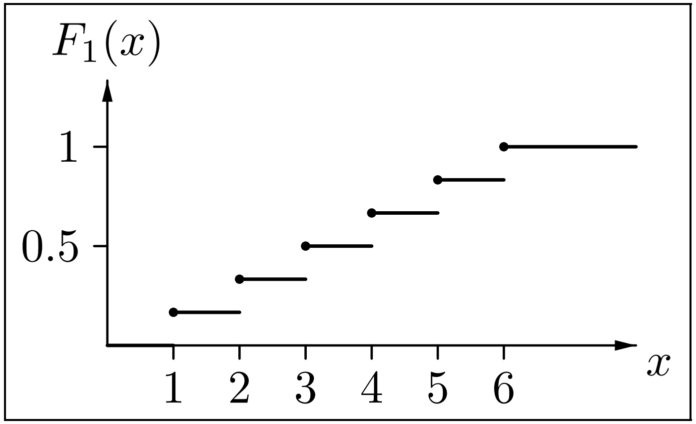
because the event of a bread roll having a weight between 80 g and 90 g is represented by the interval $[80, 90]$.

Example: Rolling a Die

$\Omega = \{1, 2, 3, 4, 5, 6\} \quad X = \text{id}$

$$p_1(\omega) = \frac{1}{6} \qquad\qquad F_1(x) = P(X \leq x)$$





$$\sum_{\omega \in \Omega} p_1(\omega) = \sum_{i=1}^{6} p_1(\omega_i) = \sum_{i=1}^{6} \frac{1}{6} = 1$$

$$P(X \leq x) = \sum_{x' \leq x} P(X = x')$$

$$P(a < X \leq b) = F_1(b) - F_1(a)$$

$$P(X = x) = P(\{X = x\}) = P(X^{-1}(x)) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$