Nonstandard Frameworks of Imprecision and Uncertainty

Content:

Random Sets

Imprecise Probabilities

Possibility Theory

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Problems with Probability Theory

Representation of Ignorance (dt. Unwissen)

We are given a die with faces $1, \ldots, 6$
What is the certainty of showing up face i ?

- Conduct a statistical survey (roll the die 10000 times) and estimate the relative frequency: $P(\{i\}) = \frac{1}{6}$
- Use subjective probabilities (which is often the normal case): We do not know anything (especially and explicitly we do not have any reason to assign unequal probabilities), so the most plausible distribution is a uniform one.

  Problem: Uniform distribution because of ignorance or extensive statistical tests

Experts analyze aircraft shapes: 3 aircraft types A, B, C
"It is type A or B with 90% certainty. About C, I don't have any clue and I do not want to commit myself. No preferences for A or B."

Problem: Propositions hard to handle with Bayesian theory

Random Sets: Modeling Imprecise Data

"$A \subseteq X$ being an imprecise date" means: the true value x_0 lies in A but there are no preferences on A.

Ω	set of possible elementary events
$\Theta = \{\xi\}$	set of observers
$\lambda(\xi)$	importance of observer ξ

Some elementary event from Ω occurs and every observer $\xi \in O$ shall announce which elementary events she personally considers possible. This set is denoted by $\Gamma(\xi) \subseteq \Omega$. $\Gamma(\xi)$ is then an imprecise date.

$\lambda : 2^\Theta \to [0, 1]$	probability measure
	(interpreted as importance measure)
$(\Theta, 2^\Theta, \lambda)$	probability space
$\Gamma : \Theta \to 2^\Omega$	set-valued mapping

Imprecise Data (2)

Let $A \subseteq \Omega$:

a) $\Gamma^*(A) \stackrel{\text{Def}}{=} \{\xi \in \Theta \mid \Gamma(\xi) \cap A \neq \emptyset\}$

b) $\Gamma_*(A) \stackrel{\text{Def}}{=} \{\xi \in \Theta \mid \Gamma(\xi) \neq \emptyset \text{ and } \Gamma(\xi) \subseteq A\}$

Remarks:

a) If $\xi \in \Gamma^*(A)$, then it is *plausible* for ξ that the occurred elementary event lies in A.

b) If $\xi \in \Gamma_*(A)$, then it is *certain* for ξ that the event lies in A.

c) $\{\xi \mid \Gamma(\xi) \neq \emptyset\} \;=\; \Gamma^*(\Omega) \;=\; \Gamma_*(\Omega)$

Let $\lambda(\Gamma^*(\Omega)) > 0$. Then we call

$$P^*(A) = \frac{\lambda(\Gamma^*(A))}{\lambda(\Gamma^*(\Omega))} \qquad \text{the upper, and} \qquad P_*(A) = \frac{\lambda(\Gamma_*(A))}{\lambda(\Gamma_*(\Omega))} \qquad \text{the lower}$$

probability w. r. t. λ and Γ.

Example

$$\Theta = \{a, b, c, d\} \qquad \lambda\colon\ a \mapsto {}^1\!/_6 \qquad \Gamma\colon\ a \mapsto \{1\}$$
$$\Omega = \{1, 2, 3\} \qquad b \mapsto {}^1\!/_6 \qquad b \mapsto \{2\}$$
$$\Gamma^*(\Omega) = \{a, b, d\} \qquad c \mapsto {}^2\!/_6 \qquad c \mapsto \emptyset$$
$$\lambda(\Gamma^*(\Omega)) = {}^4\!/_6 \qquad d \mapsto {}^2\!/_6 \qquad d \mapsto \{2, 3\}$$

A	$\Gamma^*(A)$	$\Gamma_*(A)$	$P^*(A)$	$P_*(A)$
$\emptyset$	$\emptyset$	$\emptyset$	0	0
$\{1\}$	$\{a\}$	$\{a\}$	$\frac{1}{4}$	$\frac{1}{4}$
$\{2\}$	$\{b, d\}$	$\{b\}$	$\frac{3}{4}$	$\frac{1}{4}$
$\{3\}$	$\{d\}$	$\emptyset$	$\frac{1}{2}$	0
$\{1, 2\}$	$\{a, b, d\}$	$\{a, b\}$	1	$\frac{1}{2}$
$\{1, 3\}$	$\{a, d\}$	$\{a\}$	$\frac{3}{4}$	$\frac{1}{4}$
$\{2, 3\}$	$\{b, d\}$	$\{b, d\}$	$\frac{3}{4}$	$\frac{3}{4}$
$\{1, 2, 3\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	1

One can consider $P^*(A)$ and $P_*(A)$ as upper and lower probability bounds.

Imprecise Data (3)

Some properties of probability bounds:

a) $P^*\colon\ 2^{\Omega} \rightarrow [0,1]$

b) $0 \leq P_* \leq P^* \leq 1, \qquad P_*(\emptyset) = P^*(\emptyset) = 0, \qquad P_*(\Omega) = P^*(\Omega) = 1$

c) $A \subseteq B \quad \Rightarrow \quad P^*(A) \leq P^*(B) \quad \text{and} \quad P_*(A) \leq P_*(B)$

d) $A \cap B = \emptyset \quad \not\Rightarrow \quad P^*(A) + P^*(B) = P^*(A \cup B)$

e) $P_*(A \cup B) \geq P_*(A) + P_*(B) - P_*(A \cap B)$

f) $P^*(A \cup B) \leq P^*(A) + P^*(B) - P^*(A \cap B)$

g) $P_*(A) = 1 - P^*(\Omega \backslash A)$

One can prove the following generalized equation:

$$P_*(\bigcup_{i=1}^{n} A_i) \quad \geq \sum_{\emptyset \neq I : I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot P_*(\bigcap_{i \in I} A_i)$$

These set functions also play an important role in theoretical physics (capacities, Choquet, 1955). Shafer did generalize these thoughts and developed a theory of belief functions.

Belief Revision

How is new knowledge incoporated?

Every observer announces the location of the ship in form of a subset of all possible ship locations. Given these set-valued mappings, we can derive upper and lower probabilities with the help of the observer importance measure. Let us assume the ship is certainly at sea.

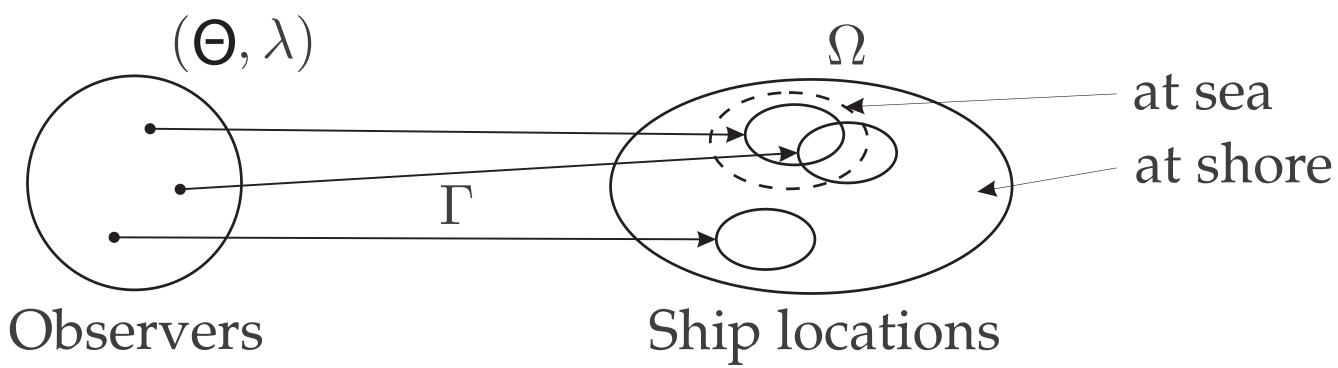
How do the upper/lower probabilities change?

Example

a) *Geometric Conditioning*
(observers that give partial or full wrong information are discarded)

$$P_*(A \mid B) = \frac{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq A \text{ and } \Gamma(\xi) \subseteq B\})}{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B\})} = \frac{P_*(A \cap B)}{P_*(B)}$$

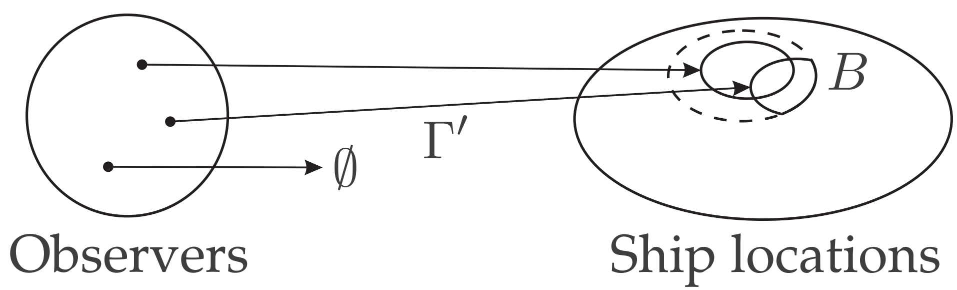
$$P^*(A \mid B) = \frac{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B \text{ and } \Gamma(\xi) \cap A \neq \emptyset\})}{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B\})} = \frac{P^*(A \cup \overline{B}) - P^*(\overline{B})}{1 - P^*(\overline{B})}$$



b) *Data Revision*
(the observed data is modified such that they fit the certain information)

$$(P_*)_B(A) = \frac{P_*(A \cup \overline{B}) - P_*(\overline{B})}{1 - P_*(B)}$$

$$(P^*)_B(A) = \frac{P^*(A \cap B)}{P^*(B)}$$



These two concepts have different semantics. There are several more belief revision concepts.

Combination of Random Sets

Let $(\Omega, 2^\Omega)$ be a space of events. Further be $(O_1, 2^{O_1}, \lambda_1)$ and $(O_2, 2^{O_2}, \lambda_2)$ spaces of independent observers.

We call $(O_1 \times O_2, \lambda_1 \cdot \lambda_2)$ the product space of observers and

$$\Gamma : O_1 \times O_2 \to 2^\Omega, \Gamma(x_1, x_2) = \Gamma_1(x_1) \cap \Gamma_2(x_2)$$

the combined observer function.

We obtain with

$$(P_L)_*(A) = \frac{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset \wedge \Gamma(x_1, x_2) \sqsubseteq A\})}{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2 \mid \Gamma(x_1, x_2) \neq \emptyset)\})}$$

the lower probability of A that respects both observations.

Example

$$\Omega = \{1,2,3\} \qquad \lambda_1\colon\ \{a\} \mapsto {}^1\!/_3 \qquad \lambda_2\colon\ \{c\} \mapsto {}^1\!/_2$$

$$\{b\} \mapsto {}^2\!/_3 \qquad \lambda_2\colon\ \{d\} \mapsto {}^1\!/_2$$

$$O_1 = \{a,b\} \qquad \Gamma_1\colon\ a \mapsto \{1,2\} \qquad \Gamma_2\colon\ c \mapsto \{1\}$$

$$O_2 = \{c,d\} \qquad b \mapsto \{2,3\} \qquad d \mapsto \{2,3\}$$

Combination:

$$O_1 \times O_2 = \{\overline{ac}, \overline{bc}, \overline{ad}, \overline{bd}\}$$

$$\begin{aligned} \lambda\colon\ \{\overline{ac}\} &\mapsto {}^1\!/_6 & \Gamma\colon\ \overline{ac} &\mapsto \{1\} & \Gamma_*(\Omega) &= \{(x_1,x_2) \mid \Gamma(x_1,x_2) \neq \emptyset\} \\ \{\overline{ad}\} &\mapsto {}^1\!/_6 & \overline{ad} &\mapsto \{2\} & &= \{\overline{ac}, \overline{ad}, \overline{bd}\} \\ \{\overline{bc}\} &\mapsto {}^2\!/_6 & \overline{bc} &\mapsto \emptyset & & \\ \{\overline{bd}\} &\mapsto {}^2\!/_6 & \overline{bd} &\mapsto \{2,3\} & \lambda(\Gamma_*(\Omega)) &= {}^4\!/_6 \end{aligned}$$

Example (2)

A	$(P_*)_{\Gamma_1}(A)$	$(P_*)_{\Gamma_2}(A)$	$(P_*)_{\Gamma}(A)$
$\emptyset$	0	0	0
$\{1\}$	0	$1/2$	$1/4$
$\{2\}$	0	0	$1/4$
$\{3\}$	0	0	0
$\{1,2\}$	$1/3$	$1/2$	$1/2$
$\{1,3\}$	0	$1/2$	$1/4$
$\{2,3\}$	$2/3$	$1/2$	$3/4$
$\{1,2,3\}$	1	1	1

Imprecise Probabilities

Let x_0 be the true value but assume there is no information about $P(A)$ to decide whether $x_0 \in A$. There are only probability boundaries.

Let $\mathcal{L}$ be a set of probability measures. Then we call

$$(P_{\mathcal{L}})_* : 2^{\Omega} \to [0,1]\,, A \mapsto \inf\{P(A) \mid P \in \mathcal{L}\} \qquad \text{the lower and}$$

$$(P_{\mathcal{L}})^* : 2^{\Omega} \to [0,1]\,, A \mapsto \sup\{P(A) \mid P \in \mathcal{L}\} \qquad \text{the upper}$$

probability of A w. r. t. $\mathcal{L}$.

a) $(P_{\mathcal{L}})_*(\emptyset) = (P_{\mathcal{L}})^*(\emptyset) = 0; \quad (P_{\mathcal{L}})_*(\Omega) = (P_{\mathcal{L}})^*(\Omega) = 1$

b) $0 \leq (P_{\mathcal{L}})_*(A) \leq (P_{\mathcal{L}})^*(A) \leq 1$

c) $(P_{\mathcal{L}})^*(A) = 1 - (P_{\mathcal{L}})_*(\overline{A})$

d) $(P_{\mathcal{L}})_*(A) + (P_{\mathcal{L}})_*(B) \leq (P_{\mathcal{L}})_*(A \cup B)$

e) $(P_{\mathcal{L}})_*(A \cap B) + (P_{\mathcal{L}})_*(A \cup B) \not\geq (P_{\mathcal{L}})_*(A) + (P_{\mathcal{L}})_*(B)$

Belief Revision

Let $B \subseteq \Omega$ and $\mathcal{L}$ a class of probabilities. The we call

$$A \subseteq \Omega : (P_{\mathcal{L}})_*(A \mid B) = \inf\{P(A \mid B) \mid P \in \mathcal{L} \wedge P(B) > 0\} \qquad \text{the lower and}$$
$$A \subseteq \Omega : (P_{\mathcal{L}})^*(A \mid B) = \sup\{P(A \mid B) \mid P \in \mathcal{L} \wedge P(B) > 0\} \qquad \text{the upper}$$

conditional probability of A given B.

A class $\mathcal{L}$ of probability measures on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is of type 1, iff there exist functions R_1 and R_2 from 2^Ω into $[0,1]$ with:

$$\mathcal{L} = \{P \mid \forall A \subseteq \Omega : R_1(A) \leq P(A) \leq R_2(A)\}$$

Belief Revision (2)

Intuition: P is determined by $P(\{\omega_i\})$, $i = 1, \ldots, n$ which corresponds to a point in $\mathbb{R}^n$ with coordinates $\big(P(\{\omega_1\}), \ldots, P(\{\omega_n\})\big)$.

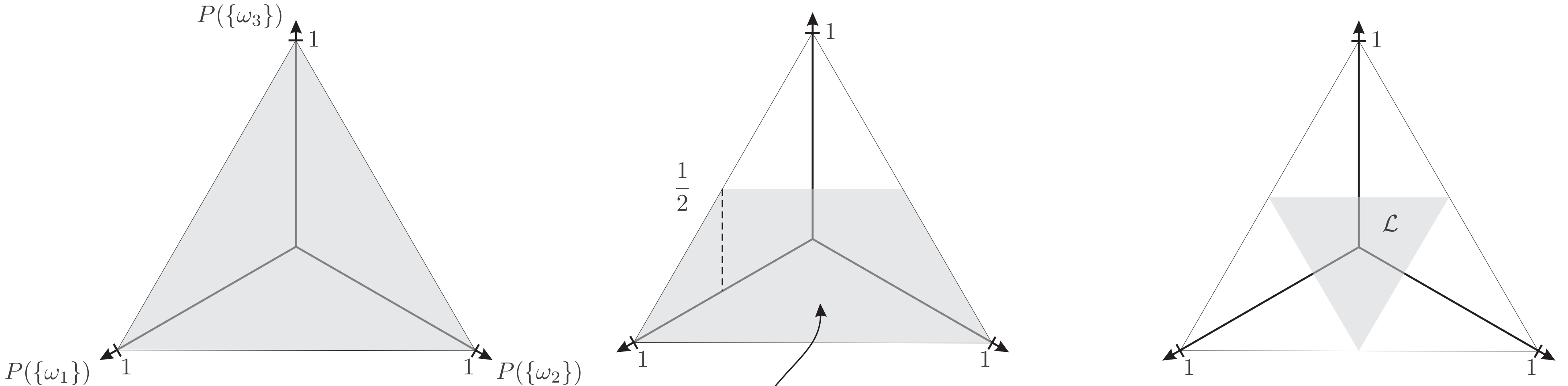
If $\mathcal{L}$ is type 1, it holds true that:

$$\mathcal{L} \Leftrightarrow \Big\{(r_1, \ldots, r_n) \in \mathbb{R}^n \mid \exists P\colon \forall A \subseteq \Omega\colon \\ (P_{\mathcal{L}})_*(A) \leq P(A) \leq (P_{\mathcal{L}})^*(A) \\ \text{and} \ \ r_i = P(\{\omega_i\}),\ i = 1, \ldots, n\Big\}$$

Example

$$\Omega = \{\omega_1, \omega_2, \omega_3\}$$

$$\mathcal{L} = \{P \mid \tfrac{1}{2} \le P(\{\omega_1, \omega_2\}) \le 1, \quad \tfrac{1}{2} \le P(\{\omega_2, \omega_3\}) \le 1, \quad \tfrac{1}{2} \le P(\{\omega_1, \omega_3\}) \le 1\}$$



general restriction:
$0 \le P(\{\omega_i\}) \le 1$
$P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_3\}) = 1$

$\{P \mid \frac{1}{2} \le P(\{\omega_1, \omega_2\}) \le 1\}$

Let $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_2, \omega_3\}$, $A_3 = \{\omega_1, \omega_3\}$

$$P_*(A_1)+P_*(A_2)+P_*(A_3)-P_*(A_1 \cap A_2)-P_*(A_2 \cap A_3)-P_*(A_1 \cap A_3)+P_*(A_1 \cap A_2 \cap A_3)$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 0 - 0 - 0 + 0 = \frac{3}{2} > 1 = P(A_1 \cup A_2 \cup A_3)$$

Belief Revision (3)

If $\mathcal{L}$ is type 1 and $(P_\mathcal{L})^*(A \cup B) \geq (P_\mathcal{L})^*(A) + (P_\mathcal{L})^*(B) - (P_\mathcal{L})^*(A \cap B)$, then

$$(P_\mathcal{L})^*(A \mid B) = \frac{(P_\mathcal{L})^*(A \cap B)}{(P_\mathcal{L})^*(A \cap B) + (P_\mathcal{L})_*(B \cap \overline{A})}$$

and

$$(P_\mathcal{L})_*(A \mid B) = \frac{(P_\mathcal{L})_*(A \cap B)}{(P_\mathcal{L})_*(A \cap B) + (P_\mathcal{L})^*(B \cap \overline{A})}$$

Let $\mathcal{L}$ be a class of type 1. $\mathcal{L}$ is of type 2, iff

$$(P_\mathcal{L})_*(A_1 \cup \dots \cup A_n) \geq \sum_{I:\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \cdot (P_\mathcal{L})_*(\bigcap_{i \in I} A_i)$$

Possibility Theory

The best-known calculus for handling uncertainty is, of course,
probability theory. [Laplace 1812]

An less well-known, but noteworthy alternative is
possibility theory. [Dubois and Prade 1988]

In the interpretation we consider here, possibility theory can handle **uncertain and imprecise information**, while probability theory, at least in its basic form, was only designed to handle *uncertain information.*

Types of **imperfect information**:

- **Imprecision:** disjunctive or set-valued information about the obtaining state, which is certain: the true state is contained in the disjunction or set.
- **Uncertainty:** precise information about the obtaining state (single case), which is not certain: the true state may differ from the stated one.
- **Vagueness:** meaning of the information is in doubt: the interpretation of the given statements about the obtaining state may depend on the user.

Possibility Theory: Axiomatic Approach

Definition: Let Ω be a (finite) sample space.
A **possibility measure** Π on Ω is a function $\Pi : 2^{\Omega} \rightarrow [0, 1]$ satisfying

1. $\Pi(\emptyset) = 0$ and
2. $\forall E_1, E_2 \subseteq \Omega : \Pi(E_1 \cup E_2) = \max\{\Pi(E_1), \Pi(E_2)\}$.

Similar to Kolmogorov's axioms of probability theory.

From the axioms follows $\Pi(E_1 \cap E_2) \leq \min\{\Pi(E_1), \Pi(E_2)\}$.

Attributes are introduced as random variables (as in probability theory).

$\Pi(A = a)$ is an abbreviation of $\Pi(\{\omega \in \Omega \mid A(\omega) = a\})$

If an event E is possible without restriction, then $\Pi(E) = 1$.
If an event E is impossible, then $\Pi(E) = 0$.

Interpretation of Degrees of Possibility [Gebhardt and Kruse 1993]

Let Ω be the (nonempty) set of all possible states of the world, ω_0 the actual (but unknown) state.

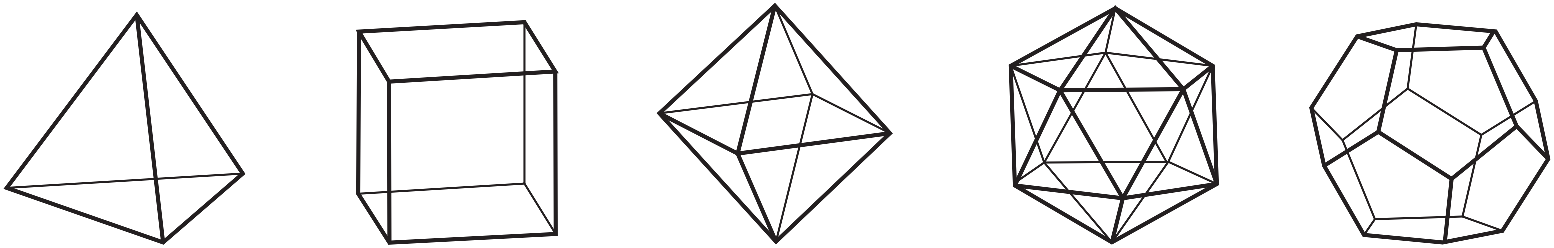
Let $C = \{c_1, \ldots, c_n\}$ be a set of contexts (observers, frame conditions etc.) and $(C, 2^C, P)$ a finite probability space (context weights).

Let $\Gamma : C \to 2^\Omega$ be a set-valued mapping, which assigns to each context the **most specific correct set-valued specification of** ω_0. The sets $\Gamma(c)$ are called the **focal sets** of Γ.

Γ is a **random set** (i.e., a set-valued random variable) [Nguyen 1978]. The **basic possibility assignment** induced by Γ is the mapping

$$\begin{aligned} \pi : \Omega &\to [0, 1] \\ \pi(\omega) &\mapsto P(\{c \in C \mid \omega \in \Gamma(c)\}). \end{aligned}$$

shaker 1	shaker 2	shaker 3	shaker 4	shaker 5



tetrahedron	hexahedron	octahedron	icosahedron	dodecahedron
1 – 4	1 – 6	1 – 8	1 – 10	1 – 12

numbers	degree of possibility
1 – 4	$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5} = 1$
5 – 6	$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5} = \frac{4}{5}$
7 – 8	$\frac{1}{5}+\frac{1}{5}+\frac{1}{5} = \frac{3}{5}$
9 – 10	$\frac{1}{5}+\frac{1}{5} = \frac{2}{5}$
11 – 12	$\frac{1}{5} = \frac{1}{5}$

From the Context Model to Possibility Measures

Definition: Let $\Gamma : C \to 2^{\Omega}$ be a random set.
The **possibility measure** induced by Γ is the mapping

$$\begin{aligned} \Pi : 2^{\Omega} &\to [0,1], \\ E &\mapsto P(\{c \in C \mid E \cap \Gamma(c) \neq \emptyset\}). \end{aligned}$$

Problem: From the given interpretation it follows only:

$$\forall E \subseteq \Omega : \quad \max_{\omega \in E} \pi(\omega) \;\leq\; \Pi(E) \;\leq\; \min\left\{1, \sum_{\omega \in E} \pi(\omega)\right\}.$$

	1	2	3	4	5
$c_1 : \frac{1}{2}$			●		
$c_2 : \frac{1}{4}$		●	●	●	
$c_3 : \frac{1}{4}$	●	●	●	●	●
π	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$

	1	2	3	4	5
$c_1 : \frac{1}{2}$			●		
$c_2 : \frac{1}{4}$	●	●			
$c_3 : \frac{1}{4}$				●	●
π	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

From the Context Model to Possibility Measures (cont.)

Attempts to solve the indicated problem:

Require the focal sets to be **consonant**:
Definition: Let $\Gamma : C \to 2^\Omega$ be a random set with $C = \{c_1, \ldots, c_n\}$. The focal sets $\Gamma(c_i)$, $1 \leq i \leq n$, are called **consonant**, iff there exists a sequence $c_{i_1}, c_{i_2}, \ldots, c_{i_n}$, $1 \leq i_1, \ldots, i_n \leq n$, $\forall 1 \leq j < k \leq n : i_j \neq i_k$, so that

$$\Gamma(c_{i_1}) \subseteq \Gamma(c_{i_2}) \subseteq \ldots \subseteq \Gamma(c_{i_n}).$$

$\rightarrow$ mass assignment theory [Baldwin *et al.* 1995]

Problem: The "voting model" is not sufficient to justify consonance.

Use the lower bound as the "most pessimistic" choice. [Gebhardt 1997]

Problem: Basic possibility assignments represent negative information, the lower bound is actually the *most optimistic* choice.

Justify the lower bound from decision making purposes.

Assume that in the end we have to decide on a single event.

Each event is described by the values of a set of attributes.

Then it can be useful to assign to a set of events the degree of possibility of the "most possible" event in the set.

Example:

$\sum$	36	18	18	28	
28	0	0	0	28	28
18	18	0	0	0	18
18	18	0	0	0	18
36	0	18	18	0	18
	18	18	18	28	max

0	40	0	40
40	0	0	40
0	0	20	20
40	40	20	max

Possibility Distributions

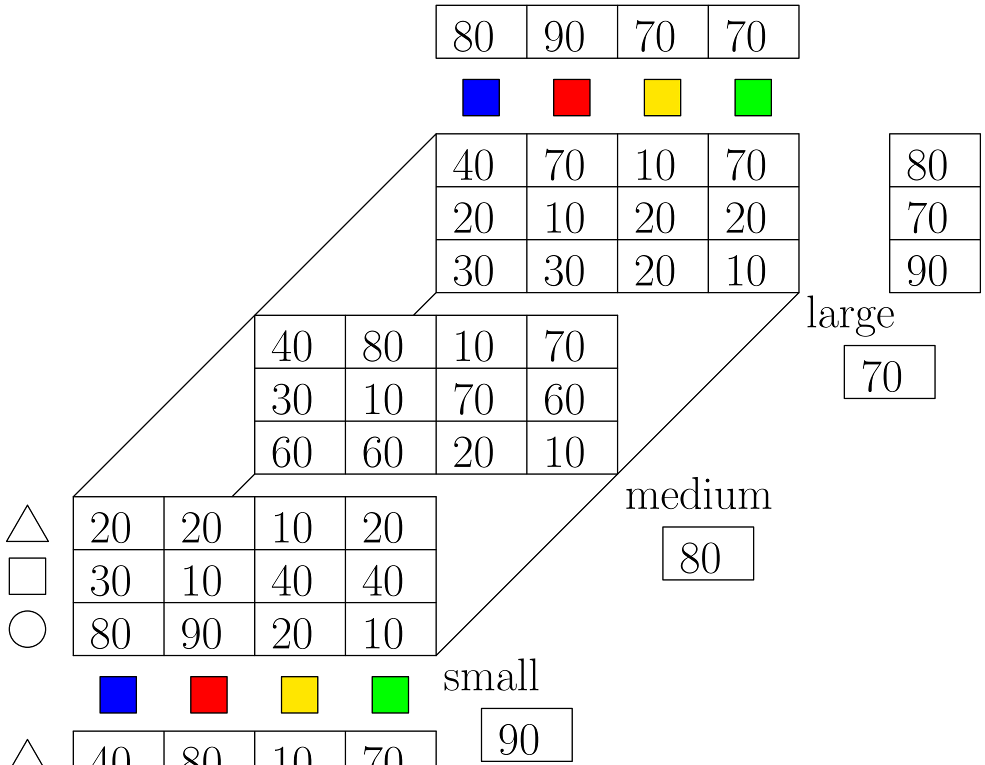
Definition: Let $X = \{A_1, \ldots, A_n\}$ be a set of attributes defined on a (finite) sample space Ω with respective domains $\mathrm{dom}(A_i)$, $i = 1, \ldots, n$. A **possibility distribution** π_X over X is the restriction of a possibility measure Π on Ω to the set of all events that can be defined by stating values for all attributes in X. That is, $\pi_X = \Pi|_{\mathcal{E}_X}$, where

$$\begin{aligned}
\mathcal{E}_X &= \Bigg\{ E \in 2^\Omega \;\Bigg|\; \exists a_1 \in \mathrm{dom}(A_1) : \ldots \exists a_n \in \mathrm{dom}(A_n) : \\
&\qquad E \mathrel{\widehat{=}} \bigwedge_{A_j \in X} A_j = a_j \Bigg\} \\
&= \Bigg\{ E \in 2^\Omega \;\Bigg|\; \exists a_1 \in \mathrm{dom}(A_1) : \ldots \exists a_n \in \mathrm{dom}(A_n) : \\
&\qquad E = \Bigg\{ \omega \in \Omega \;\Bigg|\; \bigwedge_{A_j \in X} A_j(\omega) = a_j \Bigg\} \Bigg\}.
\end{aligned}$$

Corresponds to the notion of a probability distribution.

Advantage of this formalization: No index transformation functions are needed for projections, there are just fewer terms in the conjunctions.

A Possibility Distribution



all numbers in parts per 1000

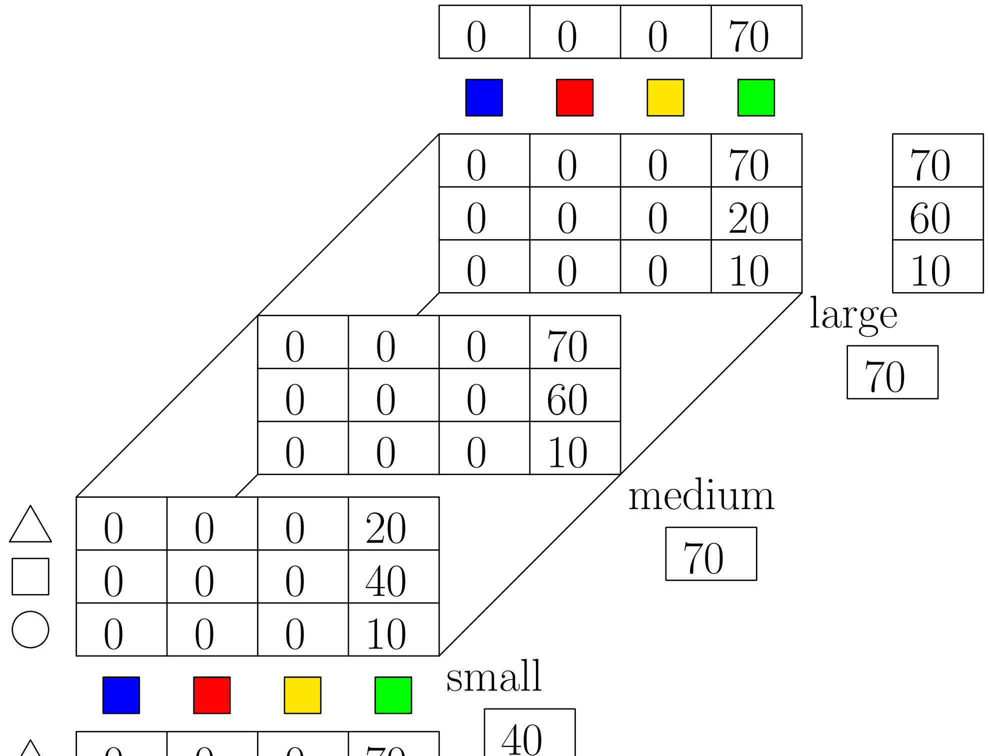
	s	m	l
△	20	80	70
□	40	70	20
○	90	60	30

	blue	red	yellow	green
△	40	80	10	70
□	30	10	70	60
○	80	90	20	10

	blue	red	yellow	green
large	40	70	20	70
medium	60	80	70	70
small	80	90	40	40

The numbers state the degrees of possibility of the corresp. value combination.

Reasoning



	blue	red	yellow	green
△	0	0	0	70
□	0	0	0	60
○	0	0	0	10

all numbers in
parts per 1000

	s	m	l
△	20	70	70
□	40	60	20
○	10	10	10

	blue	red	yellow	green
large	0	0	0	70
medium	0	0	0	70
small	0	0	0	40

Using the information that the given object is green.

Possibilistic Decomposition

As for relational and probabilistic networks, the three-dimensional possibility distribution can be decomposed into projections to subspaces, namely:

– the maximum projection to the subspace color × shape and

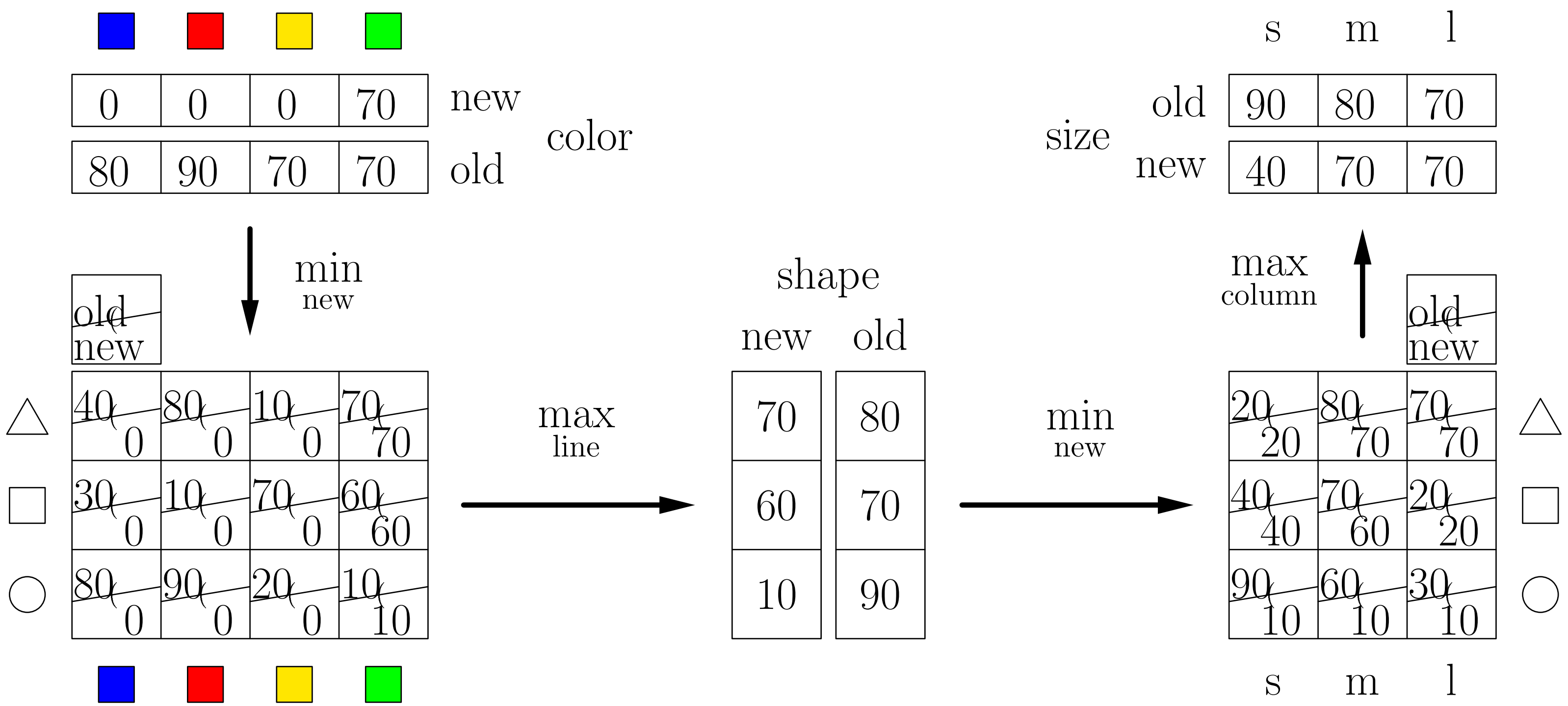
– the maximum projection to the subspace shape × size.

It can be reconstructed using the following formula:

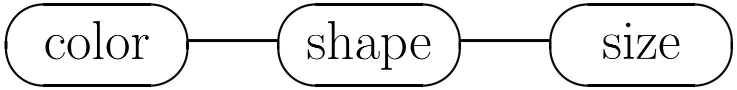
$$\begin{aligned}
\forall i,j,k:\ \pi\left(a_i^{(\text{color})}, a_j^{(\text{shape})}, a_k^{(\text{size})}\right)
&= \min\left\{\pi\left(a_i^{(\text{color})}, a_j^{(\text{shape})}\right), \pi\left(a_j^{(\text{shape})}, a_k^{(\text{size})}\right)\right\} \\
&= \min\left\{\max_k \pi\left(a_i^{(\text{color})}, a_j^{(\text{shape})}, a_k^{(\text{size})}\right),\right. \\
&\qquad\quad \left.\max_i \pi\left(a_i^{(\text{color})}, a_j^{(\text{shape})}, a_k^{(\text{size})}\right)\right\}
\end{aligned}$$

Note the analogy to the probabilistic reconstruction formulas.

Again the same result can be obtained using only projections to subspaces (maximal degrees of possibility):



This justifies a graph representation:



Conditional Possibility and Independence

Definition: Let Ω be a (finite) sample space, Π a possibility measure on Ω, and $E_1, E_2 \subseteq \Omega$ events. Then

$$\Pi(E_1 \mid E_2) = \Pi(E_1 \cap E_2)$$

is called the **conditional possibility** of E_1 given E_2.

Definition: Let Ω be a (finite) sample space, Π a possibility measure on Ω, and A, B, and C attributes with respective domains $\mathrm{dom}(A)$, $\mathrm{dom}(B)$, and $\mathrm{dom}(C)$. A and B are called **conditionally possibilistically independent** given C, written $A \perp\!\!\!\perp_\Pi B \mid C$, iff

$$\forall a \in \mathrm{dom}(A) : \forall b \in \mathrm{dom}(B) : \forall c \in \mathrm{dom}(C) :$$
$$\Pi(A = a, B = b \mid C = c) = \min\{\Pi(A = a \mid C = c), \Pi(B = b \mid C = c)\}.$$

Similar to the corresponding notions of probability theory.

Possibilistic Evidence Propagation

A:	color
B:	shape
C:	size

$$\pi(B = b \mid A = a_{\text{obs}})$$

$$= \pi\left(\bigvee_{a \in \text{dom}(A)} A = a, B = b, \bigvee_{c \in \text{dom}(C)} C = c \,\middle|\, A = a_{\text{obs}} \right)$$

$$\overset{(1)}{=} \max_{a \in \text{dom}(A)} \{ \max_{c \in \text{dom}(C)} \{ \pi(A = a, B = b, C = c \mid A = a_{\text{obs}}) \} \}$$

$$\overset{(2)}{=} \max_{a \in \text{dom}(A)} \{ \max_{c \in \text{dom}(C)} \{ \min\{ \pi(A = a, B = b, C = c), \pi(A = a \mid A = a_{\text{obs}}) \} \} \}$$

$$\overset{(3)}{=} \max_{a \in \text{dom}(A)} \{ \max_{c \in \text{dom}(C)} \{ \min\{ \pi(A = a, B = b), \pi(B = b, C = c), \pi(A = a \mid A = a_{\text{obs}}) \} \} \}$$

$$= \max_{a \in \text{dom}(A)} \{ \min\{ \pi(A = a, B = b), \pi(A = a \mid A = a_{\text{obs}}), \underbrace{\max_{c \in \text{dom}(C)} \{ \pi(B = b, C = c) \}}_{= \pi(B=b) \geq \pi(A=a, B=b)} \} \}$$

$$= \max_{a \in \text{dom}(A)} \{ \min\{ \pi(A = a, B = b), \pi(A = a \mid A = a_{\text{obs}}) \} \}$$

Belief Functions

Motivation

(Θ, Q)	Sensors	
Ω	possible results, $\Gamma : \Theta \to 2^{\Omega}$	
Γ, Q	induce a probability m on 2^{Ω}	
$m :$	$A \mapsto Q(\{\theta \in \Theta \mid \Gamma(\theta) = A\})$	mass distribution
$\mathrm{Bel} :$	$A \mapsto \sum_{B:B \subseteq A} m(B)$	Belief (lower probability)
$\mathrm{Pl} :$	$A \mapsto \sum_{B:B \cap A \neq \emptyset} m(B)$	Plausibility (upper probability)

Random sets: Dempster (1968)

Belief functions: Shafer (1974)
Development of a completely new uncertainty calculus as an alternative to Probability Theory

Belief Functions (2)

The function $\mathrm{Bel} : 2^\Omega \to [0,1]$ is called *belief function*, if it possesses the following properties:

$$\mathrm{Bel}(\emptyset) = 0$$

$$\mathrm{Bel}(\Omega) = 1$$

$$\forall n \in \mathbb{N}:\ \forall A_1, \ldots, A_n \in 2^\Omega :$$
$$\mathrm{Bel}(A_1 \cup \cdots \cup A_n) \geq \textstyle\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot \mathrm{Bel}(\bigcap_{i \in I} A_i)$$

If Bel is a belief function then for $m : 2^\Omega \to \mathbb{R}$ with $m(A) = \sum_{B:B \subseteq A} (-1)^{|A \setminus B|} \cdot \mathrm{Bel}(B)$ the following properties hold:

$$0 \leq m(A) \leq 1$$

$$m(\emptyset) = 0$$

$$\textstyle\sum_{A \subseteq \Omega} m(A) = 1$$

Let $|\Omega| < \infty$ and $f, g : 2^\Omega \to [0, 1]$.

$$\forall A \subseteq \Omega\colon \left(f(A) = \sum_{B:B\subseteq A} g(B)\right)$$

$$\Leftrightarrow$$

$$\forall A \subseteq \Omega\colon \left(g(A) = \sum_{B:B\subseteq A} (-1)^{|A\setminus B|} \cdot f(B)\right)$$

(g is called the *Möbius transformed* of f)

The mapping $m : 2^\Omega \to [0, 1]$ is called a *mass distribution*, if the following properties hold:

$m(\emptyset) = 0$

$\sum_{A\subseteq\Omega} m(A) = 1$

Example

A	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2,3\}$
$m(A)$	0	$^1/_4$	$^1/_4$	0	0	0	$^2/_4$	0
$\mathrm{Bel}(A)$	0	$^1/_4$	$^1/_4$	0	$^2/_4$	$^1/_4$	$^3/_4$	1

Belief $\widehat{=}$ lower probability with modified semantic

$$\mathrm{Bel}(\{1,3\}) = m(\emptyset) + m(\{1\}) + m(\{3\}) + m(\{1,3\})$$
$$m(\{1,3\}) = \mathrm{Bel}(\{1,3\}) - \mathrm{Bel}(\{1\}) - \mathrm{Bel}(\{3\})$$

$m(A)$	measure of the trust/belief that exactly A occurs
$\mathrm{Bel}_m(A)$	measure of total belief that A occurs
$\mathrm{Pl}_m(A)$	measure of not being able to disprove A (plausibility)

$$\mathrm{Pl}_m(A) = \sum_{B:A\cap B\neq\emptyset} m(B) = 1 - \mathrm{Bel}(\overline{A})$$

Given one of m, Bel or Pl, the other two can be efficiently computed.

$m(\Omega) = 1$, $m(A) = 0$ else	total ignorance
$m(\{\omega_0\}) = 1$, $m(A) = 0$ else	value (ω_0) known
$m(\{\omega_i\}) = p_i, \sum_{i=1}^{n} p_i = 1$	Bayesian analysis

Further intermediate steps can be modeled.

Belief Revision

Data Revision:

- Mass of A flows onto $A \cap B$.
- Masses are normalized to 1 ($\emptyset$-mass is destroyed)

Geometric Conditioning:

- Masses that do not lie completely inside B, flow off
- Normalize

The mass flow can be described by specialization matrices

Combinations of Mass Distributions

Motivation: Combination of m_1 and m_2

$m_1(A_i) \cdot m_2(B_j)$:	Mass attached to $A_i \cap B_j$, if only A_i or B_j are concerned
$\sum_{i,j:A_i \cap B_j = A} m_1(A_i) \cdot m_2(B_j)$:	Mass attached to A (after combination)

This consideration only leads to a mass distribution,
if $\sum_{i,j:A_i \cap B_j = \emptyset} m_1(A_i) \cdot m_2(B_j) = 0$.

If this sum is > 0 normalization takes place.

Combination Rule

If m_1 and m_2 are mass distributions over Ω with belief functions Bel_1 and Bel_2 and does further hold $\sum_{i,j:A_i \cap B_j = \emptyset} m_1(A_i) \cdot m_2(B_j) < 1$, then the function $m : 2^\Omega \to [0,1]\,, m(\emptyset) = 0$

$$m(A) = \frac{\sum_{B,C:B \cap C = A} m_1(B) \cdot m_2(C)}{1 - \sum_{B,C:B \cap C = \emptyset} m_1(B) \cdot m_2(C)}$$

is a mass distribution. The belief function of m is denoted as $\text{comb}(\text{Bel}_1, \text{Bel}_2)$ or $\text{Bel}_1 \oplus \text{Bel}_2$. The above formula is called the combination rule.

Example

$$m_1(\{1,2\}) = {}^1\!/_3 \qquad m_2(\{1\}) = {}^1\!/_2$$
$$m_1(\{2,3\}) = {}^2\!/_3 \qquad m_2(\{2,3\}) = {}^1\!/_2$$

$$m = m_1 \oplus m_2 :$$
$$\begin{aligned} \{1\} &\mapsto \frac{{}^1\!/_6}{{}^4\!/_6} = {}^1\!/_4 \\ \{2\} &\mapsto \frac{{}^1\!/_6}{{}^4\!/_6} = {}^1\!/_4 \\ \emptyset &\mapsto 0 \\ \{2,3\} &\mapsto \frac{{}^2\!/_6}{{}^4\!/_6} = {}^1\!/_2 \end{aligned}$$

Combination Rule (2)

Remarks:

a) The result from the combination rule and the analysis of random sets is identical

b) There are more efficient ways of combination

c) $\text{Bel}_1 \oplus \text{Bel}_2 = \text{Bel}_2 \oplus \text{Bel}_1$

d) $\oplus$ is associative

e) $\text{Bel}_1 \oplus \text{Bel}_1 \neq \text{Bel}_1$ (in general)

f) $\text{Bel}_2 : 2^\Omega \to [0,1]\,, m_2(B) = 1$

$$\text{Bel}_2(A) = \begin{cases} 1 & \text{if} B \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

The combination of Bel_1 and Bel_2 yields the data revision of m_1 with B.

Decision Making with the Pignistic Transformation

The **pignistic transformation** Bet transforms a normalized mass function m into a probability measure $P_m = Bet(m)$ as follows:

$$P_m(A) = \sum_{\emptyset \neq B \subseteq \Omega} m(B) \frac{|A \cap B|}{|B|}, \forall A \subseteq \Omega.$$

It can be shown that

$$bel(A) \leq P_m(A) \leq pl(A)$$

Decision Making - Example

There are three possible murders

Let $m(\{John\}) = 0.48, \quad m(\{John, Mary\}) = 0.12,$
$m(\{Peter, John\}) = 0.32, \quad m(\Omega) = 0.08$

We have:

$$P_m(\{John\}) = 0.48 + \frac{0.12}{2} + \frac{0.32}{2} + \frac{0.08}{3} \approx 0.73$$

$$P_m(\{Peter\}) = \frac{0.32}{2} + \frac{0.08}{3} \approx 0.19$$

$$P_m(\{Mary\}) = \frac{0.12}{2} + \frac{0.08}{3} \approx 0.09$$

The picmistic transformation givs a reasonable "Ranking"

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