Fuzzy Systems

Takagi-Sugeno Controller, Fuzzy Equivalence Relations

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Outline

Takagi-Sugeno Controller

Proposed by Tomohiro Takagi and Michio Sugeno.

Modification/extension of Mamdani controller.

Both in common: fuzzy partitions of input domain $X_1, \ldots, X_n$.

Difference to Mamdani controller:

- no fuzzy partition of output domain Y,
- controller rules $R_1, \ldots, R_k$ are given by

$$R_r : \textbf{if } \xi_1 \text{ is } A^{(1)}_{i_{1,r}} \text{ and } \ldots \text{ and } \xi_n \text{ is } A^{(n)}_{i_{n,r}}$$
$$\textbf{then } \eta_r = f_r(\xi_1, \ldots, \xi_n),$$

  $f_r : X_1 \times \ldots \times X_n \to Y$.
- Generally, f_r is linear, *i.e.* $f_r(x_1, \ldots, x_n) = a_0^{(r)} + \sum_{i=1}^{n} a_i^{(r)} x_i$.

Takagi-Sugeno Controller: Conclusion

For given input $(x_1, \ldots, x_n)$ and for each R_r, decision logic computes truth value α_r of each premise, and then $f_r(x_1, \ldots, x_n)$.

Analogously to Mamdani controller:

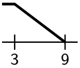
$$\alpha_r = \min\left\{\mu^{(1)}_{i_{1,r}}(x_1), \ldots, \mu^{(n)}_{i_{n,r}}(x_n)\right\}.$$

Output equals crisp control value

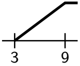
$$\eta = \frac{\sum_{r=1}^{k} \alpha_r \cdot f_r(x_1, \ldots, x_n)}{\sum_{r=1}^{k} \alpha_r}.$$

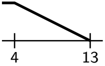
Thus no defuzzification method necessary.

Example

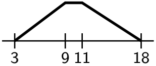
R_1 : if ξ_1 is 

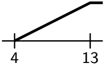
then $\eta_1 = 1 \cdot \xi_1 + 0.5 \cdot \xi_2 + 1$

R_2 : if ξ_1 is 

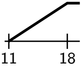
and ξ_2 is 

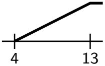
then $\eta_2 = -0.1 \cdot \xi_1 + 4 \cdot \xi_2 + 1.2$

R_3 : if ξ_1 is 

and ξ_2 is 

then $\eta_3 = 0.9 \cdot \xi_1 + 0.7 \cdot \xi_2 + 9$

R_4 : if ξ_1 is 

and ξ_2 is 

then $\eta_4 = 0.2 \cdot \xi_1 + 0.1 \cdot \xi_2 + 0.2$

If a certain clause "x_j is $A^{(j)}_{i_{j,r}}$" in rule R_r is missing,
then $\mu_{i_{j,r}}(x_j) \equiv 1$ for all linguistic values $i_{j,r}$.

For instance, here x_2 in R_1, so $\mu_{i_{2,1}}(x_2) \equiv 1$ for all $i_{2,1}$.

Example: Output Computation

input: $(\xi_1, \xi_2) = (6, 7)$

$$\alpha_1 = 1/2 \wedge 1 = 1/2 \qquad \eta_1 = 6 + 7/2 + 1 = 10.5$$

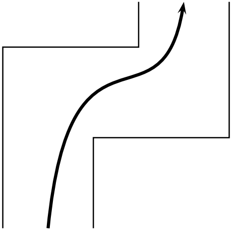
$$\alpha_2 = 1/2 \wedge 2/3 = 1/2 \qquad \eta_2 = -0.6 + 28 + 1.2 = 28.6$$

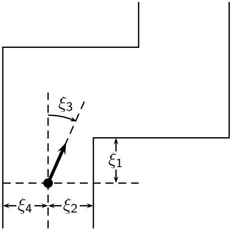
$$\alpha_3 = 1/2 \wedge 1/3 = 1/3 \qquad \eta_3 = 0.9 \cdot 6 + 0.7 \cdot 7 + 9 = 19.3$$

$$\alpha_4 = 0 \wedge 1/3 = 0 \qquad \eta_4 = 6 + 7/2 + 1 = 10.5$$

output: $\eta = f(6, 7) = \dfrac{1/2 \cdot 10.5 + 1/2 \cdot 28.6 + 1/3 \cdot 19.3}{1/2 + 1/2 + 1/3} = 19.5$

Example: Passing a Bend





Pass a bend with a car at constant speed.

Measured inputs:

ξ_1 : distance of car to beginning of bend

ξ_2 : distance of car to inner barrier

ξ_3 : direction (angle) of car

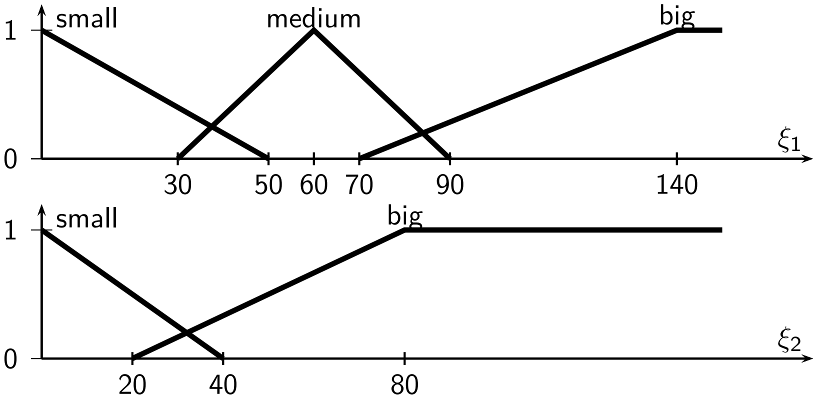
ξ_4 : distance of car to outer barrier

η = rotation speed of steering wheel

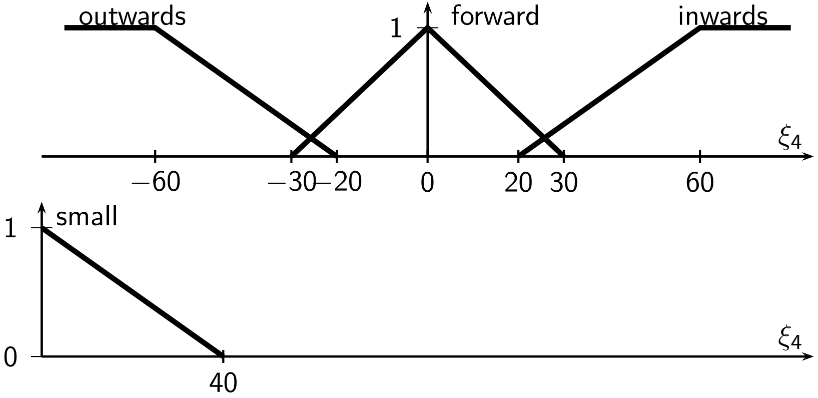
$X_1 = [0\,\text{cm}, 150\,\text{cm}]$, $X_2 = [0\,\text{cm}, 150\,\text{cm}]$

$X_3 = [-90\,°, 90\,°]$, $X_4 = [0\,\text{cm}, 150\,\text{cm}]$

Fuzzy Partitions of X_1 and X_2



Fuzzy Partitions of X_3 and X_4



Form of Rules of Car

$$R_r : \textbf{if } \xi_1 \text{ is } A \text{ and } \xi_2 \text{ is } B \text{ and } \xi_3 \text{ is } C \text{ and } \xi_4 \text{ is } D$$

$$\textbf{then } \eta = p_0^{(A,B,C,D)} + p_1^{(A,B,C,D)} \cdot \xi_1 + p_2^{(A,B,C,D)} \cdot \xi_2 + p_3^{(A,B,C,D)} \cdot \xi_3 + p_4^{(A,B,C,D)} \cdot \xi_4$$

$A \in \{small,\ medium,\ big\}$

$B \in \{small,\ big\}$

$C \in \{outwards,\ forward,\ inwards\}$

$D \in \{small\}$

$p_0^{(A,B,C,D)}, \ldots, p_4^{(A,B,C,D)} \in \mathbb{R}$

Control Rules for the Car

rule	ξ_1	ξ_2	ξ_3	ξ_4	p_0	p_1	p_2	p_3	p_4
R_1	-	-	outwards	small	3.000	0.000	0.000	−0.045	−0.004
R_2	-	-	forward	small	3.000	0.000	0.000	−0.030	−0.090
R_3	small	small	outwards	-	3.000	−0.041	0.004	0.000	0.000
R_4	small	small	forward	-	0.303	−0.026	0.061	−0.050	0.000
R_5	small	small	inwards	-	0.000	−0.025	0.070	−0.075	0.000
R_6	small	big	outwards	-	3.000	−0.066	0.000	−0.034	0.000
R_7	small	big	forward	-	2.990	−0.017	0.000	−0.021	0.000
R_8	small	big	inwards	-	1.500	0.025	0.000	−0.050	0.000
R_9	medium	small	outwards	-	3.000	−0.017	0.005	−0.036	0.000
R_{10}	medium	small	forward	-	0.053	−0.038	0.080	−0.034	0.000
R_{11}	medium	small	inwards	-	−1.220	−0.016	0.047	−0.018	0.000
R_{12}	medium	big	outwards	-	3.000	−0.027	0.000	−0.044	0.000
R_{13}	medium	big	forward	-	7.000	−0.049	0.000	−0.041	0.000
R_{14}	medium	big	inwards	-	4.000	−0.025	0.000	−0.100	0.000
R_{15}	big	small	outwards	-	0.370	0.000	0.000	−0.007	0.000
R_{16}	big	small	forward	-	−0.900	0.000	0.034	−0.030	0.000
R_{17}	big	small	inwards	-	−1.500	0.000	0.005	−0.100	0.000
R_{18}	big	big	outwards	-	1.000	0.000	0.000	−0.013	0.000
R_{19}	big	big	forward	-	0.000	0.000	0.000	−0.006	0.000
R_{20}	big	big	inwards	-	0.000	0.000	0.000	−0.010	0.000

Sample Calculation

Assume that the car is 10 cm away from beginning of bend ($\xi_1 = 10$).

The distance of the car to the inner barrier be 30 cm ($\xi_2 = 30$).

The distance of the car to the outer barrier be 50 cm ($\xi_4 = 50$).

The direction of the car be "forward" ($\xi_3 = 0$).

Then according to all rules $R_1, \ldots, R_{20}$,
only premises of R_4 and R_7 have a value $\neq 0$.

Membership Degrees to Control Car

	small	medium	big
$\xi_1 = 10$	0.8	0	0

	small	big
$\xi_2 = 30$	0.25	0.167

	outwards	forward	inwards
$\xi_3 = 0$	0	1	0

	small
$\xi_4 = 50$	0

Sample Calculation (cont.)

For the premise of R_4 and R_7, $\alpha_4 = 1/4$ and $\alpha_7 = 1/6$, resp.

The rules weights $\alpha_4 = \frac{1/4}{1/4+1/6} = 3/5$ for R_4 and $\alpha_5 = 2/5$ for R_7.

R_4 yields

$$\begin{aligned}\eta_4 &= 0.303 - 0.026 \cdot 10 + 0.061 \cdot 30 - 0.050 \cdot 0 + 0.000 \cdot 50 \\ &= 1.873.\end{aligned}$$

R_7 yields

$$\begin{aligned}\eta_4 &= 2.990 - 0.017 \cdot 10 + 0.000 \cdot 30 - 0.021 \cdot 0 + 0.000 \cdot 50 \\ &= 2.820.\end{aligned}$$

The final value for control variable is thus

$$\eta = 3/5 \cdot 1.873 + 2/5 \cdot 2.820 = 2.2518.$$

Outline

Interpolation in the Presence of Fuzziness

Both Takagi-Sugeno and Mamdani are based on heuristics.

They are used without a concrete interpretation.

Fuzzy control is interpreted as a method to specify a non-linear transition function by knowledge-based interpolation.

A fuzzy controller can be interpreted as fuzzy interpolation.

Now recall the concept of **fuzzy equivalence relations** (also called **similarity relations**).

Similarity: An Example

Specification of a partial control mapping (“good control actions”):

		gradient						
		-40.0	-6.0	-3.0	0.0	3.0	6.0	40.0
	-70.0	22.5	15.0	15.0	10.0	10.0	5.0	5.0
	-50.0	22.5	15.0	10.0	10.0	5.0	5.0	0.0
	-30.0	15.0	10.0	5.0	5.0	0.0	0.0	0.0
deviation	0.0	5.0	5.0	0.0	0.0	0.0	-10.0	-15.0
	30.0	0.0	0.0	0.0	-5.0	-5.0	-10.0	-10.0
	50.0	0.0	-5.0	-5.0	-10.0	-15.0	-15.0	-22.5
	70.0	-5.0	-5.0	-15.0	-15.0	-15.0	-15.0	-15.0

Interpolation of Control Table

There might be additional knowledge available:

Some values are "indistinguishable", "similar" or "approximately equal".

Or they should be treated in a similar way.

Two problems:

a) How to model information about similarity?

b) How to interpolate in case of an existing similarity information?

How to Model Similarity?

Proposal 1: Equivalence Relation

Definition

Let A be a set and $\approx$ be a binary relation on A. $\approx$ is called an equivalence relation if and only if $\forall a, b, c \in A$,

(i)	$a \approx a$	(reflexivity)
(ii)	$a \approx b \leftrightarrow b \approx a$	(symmetry)
(iii)	$a \approx b \wedge b \approx c \rightarrow a \approx c$	(transitivity).

Let us try $a \approx b \Leftrightarrow |a - b| < \varepsilon$ where ε is fixed.

$\approx$ is not transitive, $\approx$ is no equivalence relation.

Recall the Poincaré paradox: $a \approx b,\ b \approx c,\ a \not\approx c$.

This is counterintuitive.

How to Model Similarity?

Proposal 2: Fuzzy Equivalence Relation

Definition

A function $E : X^2 \to [0, 1]$ is called a fuzzy equivalence relation with respect to the t-norm $\top$ if it satisfies the following conditions

$$\forall x, y, z \in X \quad \begin{array}{lll} (i) & E(x, x) = 1 & \text{(reflexivity)} \\ (ii) & E(x, y) = E(y, x) & \text{(symmetry)} \\ (iii) & \top(E(x, y), E(y, z)) \leq E(x, z) & (t\text{-transitivity}). \end{array}$$

$E(x, y)$ is the degree to which $x \approx y$ holds.

E is also called similarity relation, t-equivalence relation, indistinguishability operator, or tolerance relation.

Note that property (*iii*) corresponds to the vague statement if $(x \approx y) \wedge (y \approx z)$ then $x \approx z$.

Fuzzy Equivalence Relations: An Example

Let δ be a pseudo metric on X.

Furthermore $\top(a, b) = \max\{a + b - 1,\ 0\}$ Łukasiewicz t-norm.

Then $E_\delta(x, y) = 1 - \min\{\delta(x, y),\ 1\}$ is a fuzzy equivalence relation.

$\delta(x, y) = 1 - E_\delta(x, y)$ is the induced pseudo metric.

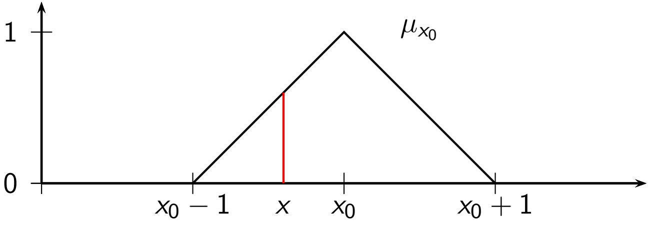
Here, fuzzy equivalence and distance are dual notions in this case.

Definition
A function $E : X^2 \to [0, 1]$ is called a fuzzy equivalence relation if
$\forall x, y, z \in X$

(i)	$E(x, x) = 1$	(reflexivity)
(ii)	$E(x, y) = E(y, x)$	(symmetry)
(iii)	$\max\{E(x, y) + E(y, z) - 1,\ 0\} \leq E(x, z)$	(Łukasiewicz transitivity).

Fuzzy Sets as Derived Concept

$\delta(x,y) = \|x-y\|$	metric
$E_\delta(x,y) = 1 - \min\{\|x-y\|, 1\}$	fuzzy equivalence relation



$$\mu_{x_0} : X \to [0,1]$$
$$x \mapsto E_\delta(x, x_0) \quad \text{fuzzy singleton}$$

μ_{x_0} describes "local" similarities.

Extensional Hull

$E : \mathbb{R} \times \mathbb{R} \to [0,1], \quad (x,y) \mapsto 1 - \min\{|x-y|,\ 1\}$ is fuzzy equivalence relation *w.r.t.* $\top_{\text{Łuka}}$.

Definition
Let E be a fuzzy equivalence relation on X *w.r.t.* $\top$.

$\mu \in \mathcal{F}(X)$ is extensional if and only if
$\forall x, y \in X : \top(\mu(x), E(x,y)) \leq \mu(y)$.

Definition
Let E be a fuzzy equivalence relation on a set X.

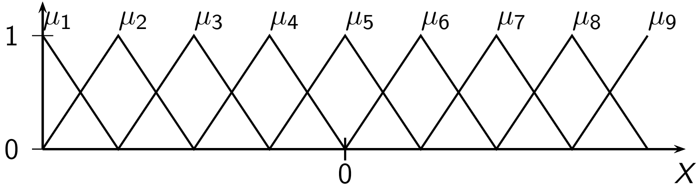
Then the extensional hull of a set $M \subseteq X$ is the fuzzy set

$$\mu_M : X \to [0,1], \qquad x \mapsto \sup\{E(x,y) \mid y \in M\}.$$

The extensional hull of $\{x_0\}$ is called a singleton.

Specification of Fuzzy Equivalence Relation

Given a family of fuzzy sets that describes "local" similarities.



There exists a fuzzy equivalence relation on X with induced singletons μ_i if and only if

$$\forall i, j : \sup_{x \in X}\{\mu_i(x) + \mu_j(x) - 1\} \leq \inf_{y \in X}\{1 - |\mu_i(y) - \mu_j(y)|\}.$$

If $\mu_i(x) + \mu_j(x) \leq 1$ for $i \neq j$, then there is a fuzzy equivalence relation E on X

$$E(x, y) = \inf_{i \in I}\{1 - |\mu_i(x) - \mu_i(y)|\}.$$

Necessity of Scaling I

Are there other fuzzy equivalence relations on $\mathbb{R}$ than $E(x, y) = 1 - \min\{|x - y|,\ 1\}$?

Integration of scaling.

A fuzzy equivalence relation depends on the measurement unit, *e.g.*

- Celsius: $E(20\,^\circ\mathrm{C}, 20.5\,^\circ\mathrm{C}) = 0.5$,
- Fahrenheit: $E(68\,\mathrm{F}, 68.9\,\mathrm{F}) = 0.5$,
- scaling factor for Celsius/Fahrenheit $= 1.8$ $(F = 9/5C + 32)$.

$E(x, y) = 1 - \min\{|c \cdot x - c \cdot y|,\ 1\}$ is a fuzzy equivalence relation!

Necessity of Scaling II

How to generalize scaling concept?

$X = [a, b]$.

Scaling $c : X \to [0, \infty)$.

Transformation

$$f : X \to [0, \infty), \quad x \mapsto \int_a^x c(t)dt.$$

Fuzzy equivalence relation

$$E : X \times X \to [0, 1], \quad (x, y) \mapsto 1 - \min\{|f(x) - f(y)|,\ 1\}.$$

Fuzzy Equivalence Relations: Fuzzy Control

The imprecision of measurements is modeled by a fuzzy equivalence relations $E_1, \dots, E_n$ and F on $X_1, \dots, X_n$ and Y, resp.

The information provided by control expert are

- k input-output tuples $(x_1^{(r)}, \dots, x_n^{(r)}, y^{(r)})$ and
- the description of the fuzzy equivalence relations for input and output spaces, resp.

The goal is to derive a control function $\varphi : X_1 \times \dots \times X_n \to Y$ from this information.

Determine Fuzzy-valued Control Functions I

The extensional hull of graph of φ must be determined.

Then the equivalence relation on $X_1 \times \ldots \times X_n \times Y$ is

$$E((x_1, \ldots, x_n, y),\ (x_1', \ldots, x_n', y')) \\ = \min\{E_1(x_1, x_1'), \ldots, E_n(x_n, x_n'), F(y, y')\}.$$

Determine Fuzzy-valued Control Functions II

For X_i and Y, define the sets

$$X_i^{(0)} = \left\{x \in X_i \mid \exists r \in \{1, \ldots, k\} : x = x_i^{(r)}\right\}$$

and

$$Y^{(0)} = \left\{y \in Y \mid \exists r \in \{1, \ldots, k\} : y = y^{(r)}\right\}.$$

$X_i^{(0)}$ and $Y^{(0)}$ contain all values of the r input-output tuples $(x_1^{(r)}, \ldots, x_n^{(r)}, y^{(r)})$.

For each $x_0 \in X_i^{(0)}$, singleton μ_{x_0} is obtained by

$$\mu_{x_0}(x) = E_i(x, x_0).$$

Determine Fuzzy-valued Control Functions III

If φ is only partly given, then use $E_1, \ldots, E_n, F$ to fill the gaps of φ_0.

The extensional hull of φ_0 is a fuzzy set

$$\mu_{\varphi_0}(x'_1, \ldots, x'_n, y') = \max_{r \in \{1,\ldots,k\}} \left\{ \min\{E_1(x_1^{(r)}, x'_1), \ldots, E_n(x_n^{(r)}, x'_n), F(y^{(r)}, y')\} \right\}.$$

μ_{φ_0} is the smallest fuzzy set containing the graph of φ_0.

Obviously, $\mu_{\varphi_0} \leq \mu_\varphi$

$$\mu_{\varphi_0}^{(x_1,\ldots,x_n)} : Y \to [0,1], \quad y \mapsto \mu_{\varphi_0}(x_1, \ldots, x_n, y).$$

Reinterpretation of Mamdani Controller

For input $(x_1, \ldots, x_n)$, the projection of the extensional hull of graph of φ_0 leads to a fuzzy set as output.

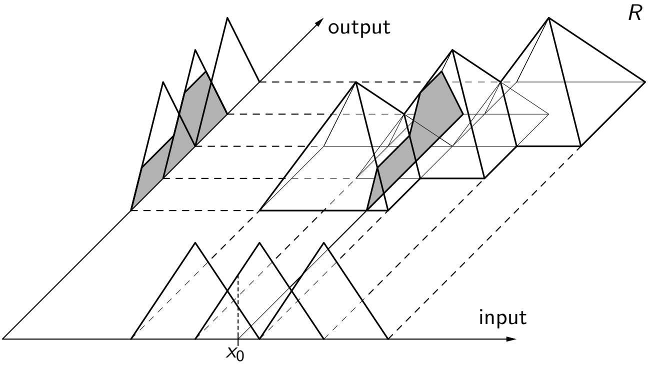
This is identical to the Mamdani controller output.

It identifies the input-output tuples of φ_0 by linguistic rules:

$$\begin{aligned} R_r : & \textbf{ if } \mathcal{X}_1 \textbf{ is } \textit{approximately } x_1^{(r)} \\ & \textbf{and} \ldots \\ & \textbf{and } \mathcal{X}_n \textbf{ is } \textit{approximately } x_n^{(r)} \\ & \textbf{then } \mathcal{Y} \text{ is } y^{(r)}. \end{aligned}$$

A fuzzy controller based on equivalence relations behaves like a Mamdani controller.

Reinterpretation of Mamdani Controller



3 fuzzy rules (specified by 3 input-output tuples).

The extensional hull is the maximum of all fuzzy rules.

Literature about Fuzzy Control

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