Elements of Graph Theory

Simple Graph

A simple graph (or just: graph) is a tuple $\mathcal{G} = (V, E)$ where

$$V = \{A_1, \ldots, A_n\}$$

represents a finite set of **vertices** (or **nodes**) and

$$E \subseteq (V \times V) \setminus \{(A, A) \mid A \in V\}$$

denotes the set of **edges**.
It is called simple since there are no self-loops and no multiple edges.

Edge Types

Let $\mathcal{G} = (V, E)$ be a graph. An edge $e = (A, B)$ is called

- **directed** if $(A, B) \in E \;\Rightarrow\; (B, A) \notin E$
  Notation: $A \to B$
- **undirected** if $(A, B) \in E \;\Rightarrow\; (B, A) \in E$
  Notation: $A - B$ or $B - A$

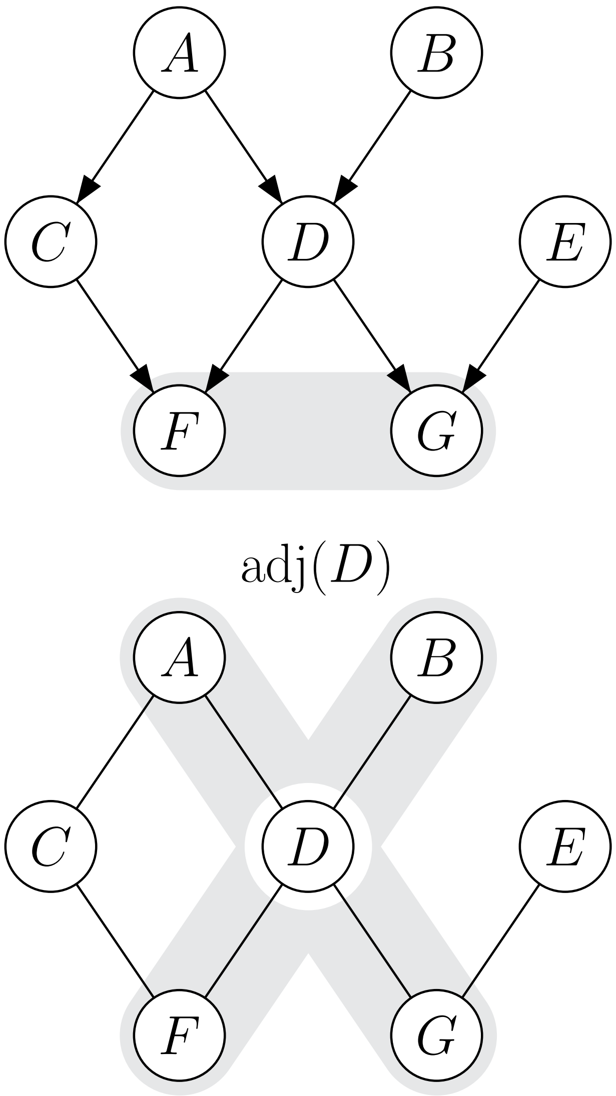
(Un)directed Graph

A graph with only (un)directed edges is called an (un)directed graph.

Adjacency Set

Let $\mathcal{G} = (V, E)$ be a graph. The set of nodes that is accessible via a given node $A \in V$ is called the **adjacency set** of A:

$$\mathrm{adj}(A) \;=\; \{B \in V \mid (A, B) \in E\}$$



Paths

Let $\mathcal{G} = (V, E)$ be a graph. A series ρ of r pairwise different nodes

$$\rho = \left\langle A_{i_1}, \ldots, A_{i_r} \right\rangle$$

is called a **path** from A_i to A_j if

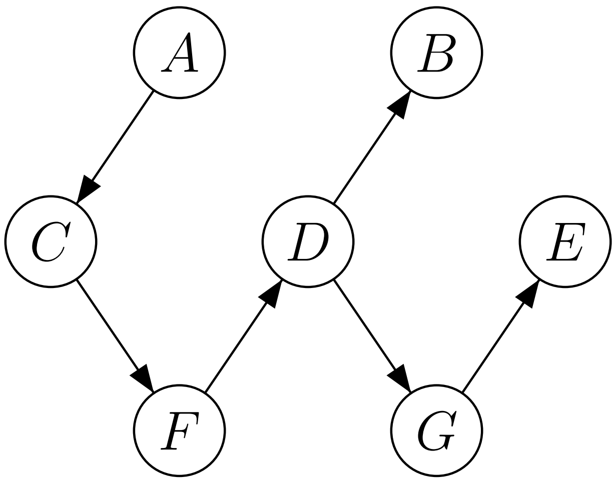
- $A_{i_1} = A_i, \quad A_{i_r} = A_j$
- $A_{i_{k+1}} \in \operatorname{adj}(A_{i_k}), \quad 1 \le k < r$

A path with only undirected edges is called an **undirected path**

$$\rho = A_{i_1} - \cdots - A_{i_r}$$

whereas a path with only directed edges is referred to as a **directed path**

$$\rho = A_{i_1} \to \cdots \to A_{i_r}$$



If there is a directed path ρ from node A to node B in a directed graph $\mathcal{G}$ we write

$$A \overset{\rho}{\underset{\mathcal{G}}{\rightsquigarrow}} B.$$

If the path ρ is undirected we denote this with

$$A \overset{\rho}{\underset{\mathcal{G}}{\leftrightsquigarrow}} B.$$

Loop

Let $\mathcal{G} = (V, E)$ be an undirected graph. A path

$$\rho = X_1 - \cdots - X_k$$

with $X_k - X_1 \in E$ is called a loop.

Cycle

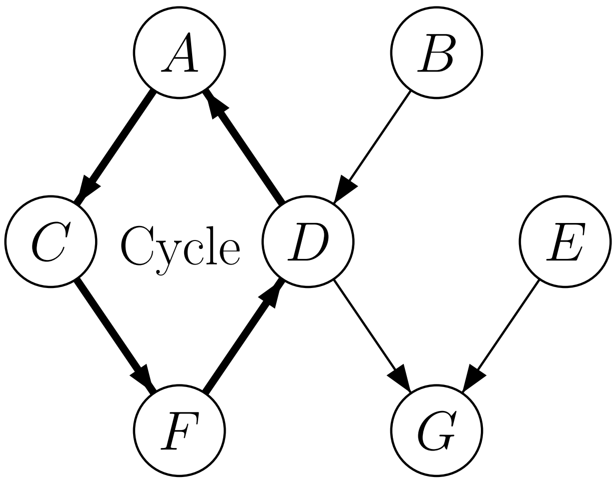
Let $\mathcal{G} = (V, E)$ be a directed graph. A path

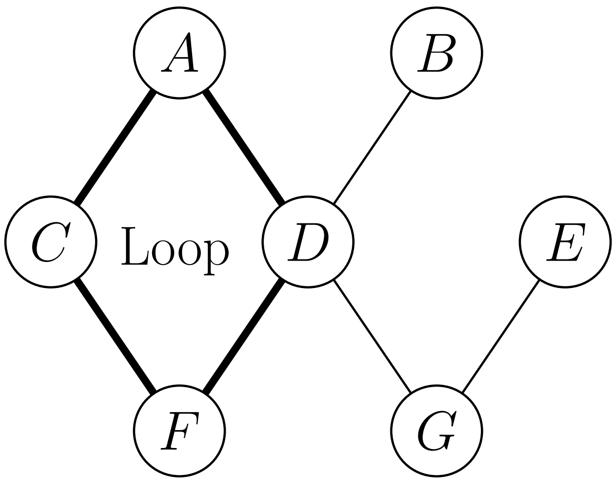
$$\rho = X_1 \to \cdots \to X_k$$

with $X_k \to X_1 \in E$ is called a cycle.

Directed Acyclic Graph (DAG)

A directed graph $\mathcal{G} = (V, E)$ is called **acyclic** if for every path $X_1 \to \cdots \to X_k$ in $\mathcal{G}$ the condition $X_k \to X_1 \notin E$ is satisfied, i.e. it contains no cycle.





Parents, Children and Families

Let $\mathcal{G} = (V, E)$ be a directed graph. For every node $A \in V$ we define the following sets:

- **Parents:**

  $$\text{parents}_{\mathcal{G}}(A) = \{B \in V \mid B \to A \in E\}$$

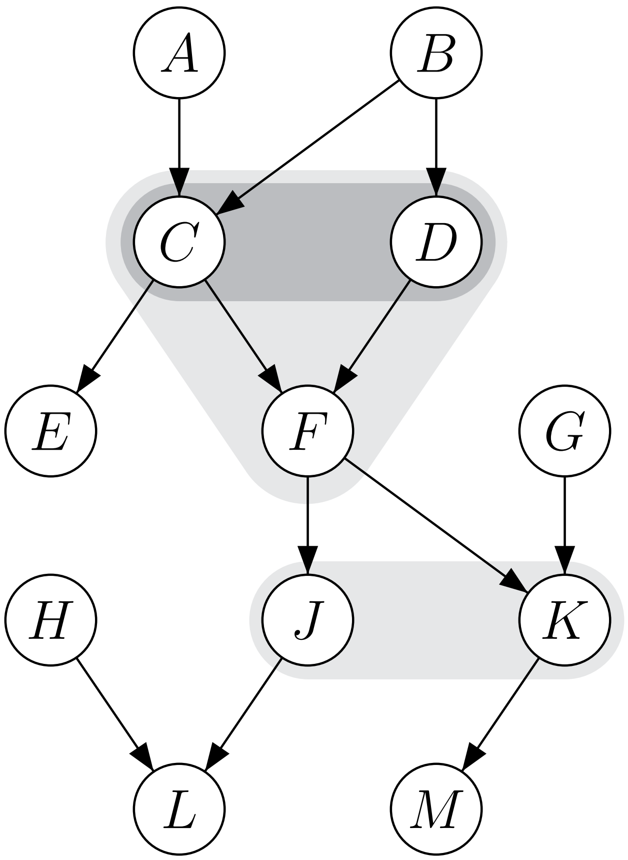
- **Children:**

  $$\text{children}_{\mathcal{G}}(A) = \{B \in V \mid A \to B \in E\}$$

- **Family:**

  $$\text{family}_{\mathcal{G}}(A) = \{A\} \cup \text{parents}_{\mathcal{G}}(A)$$

If the respective graph is clear from the context, the index $\mathcal{G}$ is omitted.



$$\begin{aligned} \text{parents}(F) &= \{C, D\} \\ \text{children}(F) &= \{J, K\} \\ \text{family}(F) &= \{C, D, F\} \end{aligned}$$

Let $\mathcal{G} = (V, E)$ be a DAG. For every node $A \in V$ we define the following sets:

- **Ancestors:**

$$\text{ancs}_{\mathcal{G}}(A) = \{B \in V \mid \exists \rho : B \overset{\rho}{\underset{\mathcal{G}}{\leadsto}} A\}$$

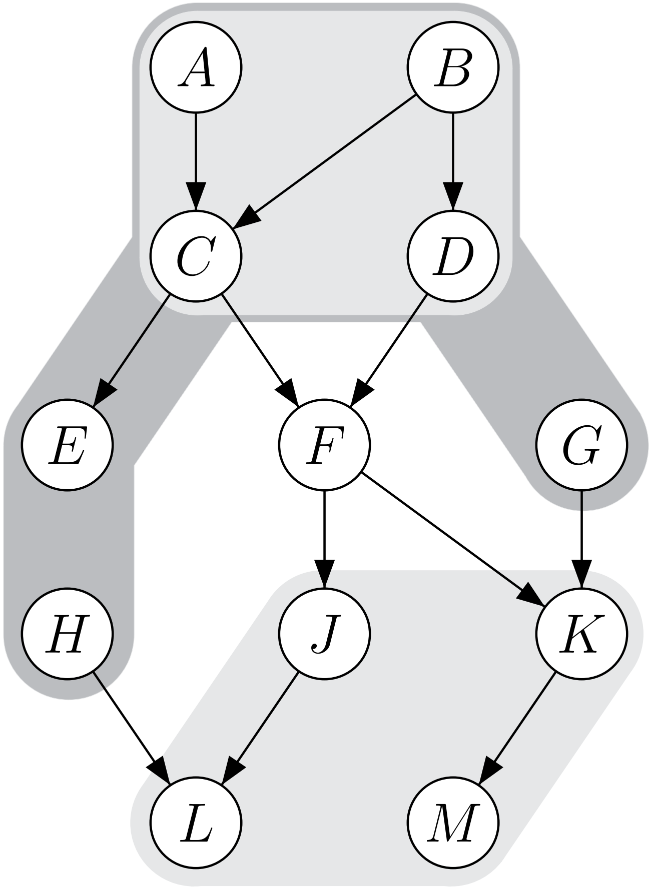
- **Descendants:**

$$\text{descs}_{\mathcal{G}}(A) = \{B \in V \mid \exists \rho : A \overset{\rho}{\underset{\mathcal{G}}{\leadsto}} B\}$$

- **Non-Descendants:**

$$\text{non-descs}_{\mathcal{G}}(A) = V \setminus \{A\} \setminus \text{descs}_{\mathcal{G}}(A)$$

If the respective graph is clear from the context, the index $\mathcal{G}$ is omitted.



$$\text{ancs}(F) = \{A, B, C, D\}$$
$$\text{descs}(F) = \{J, K, L, M\}$$
$$\text{non-descs}(F) = \{A, B, C, D, E, G, H\}$$

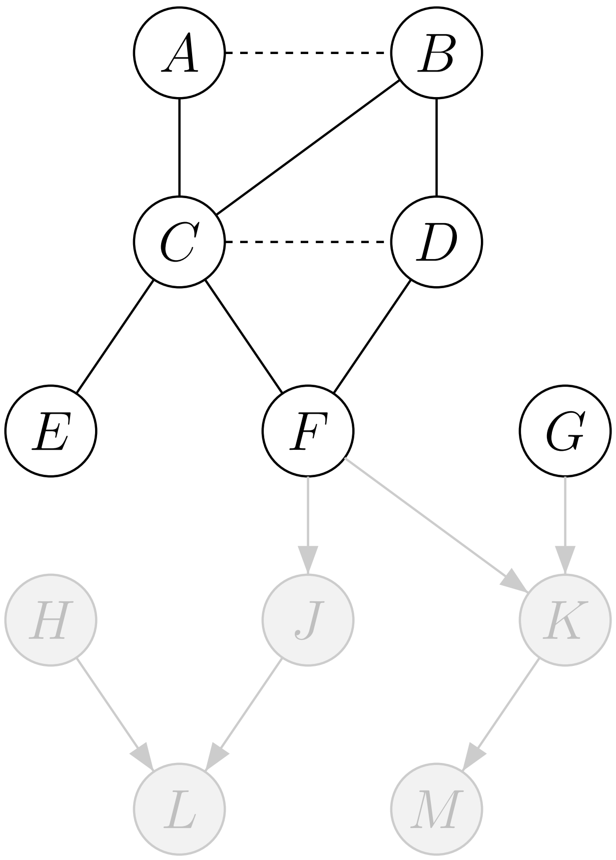
Operations on Graphs

Let $\mathcal{G} = (V, E)$ be a DAG.

The **Minimal Ancestral Subgraph** of $\mathcal{G}$ given a set $M \subseteq V$ of nodes is the smallest subgraph that contains all ancestors of all nodes in M.

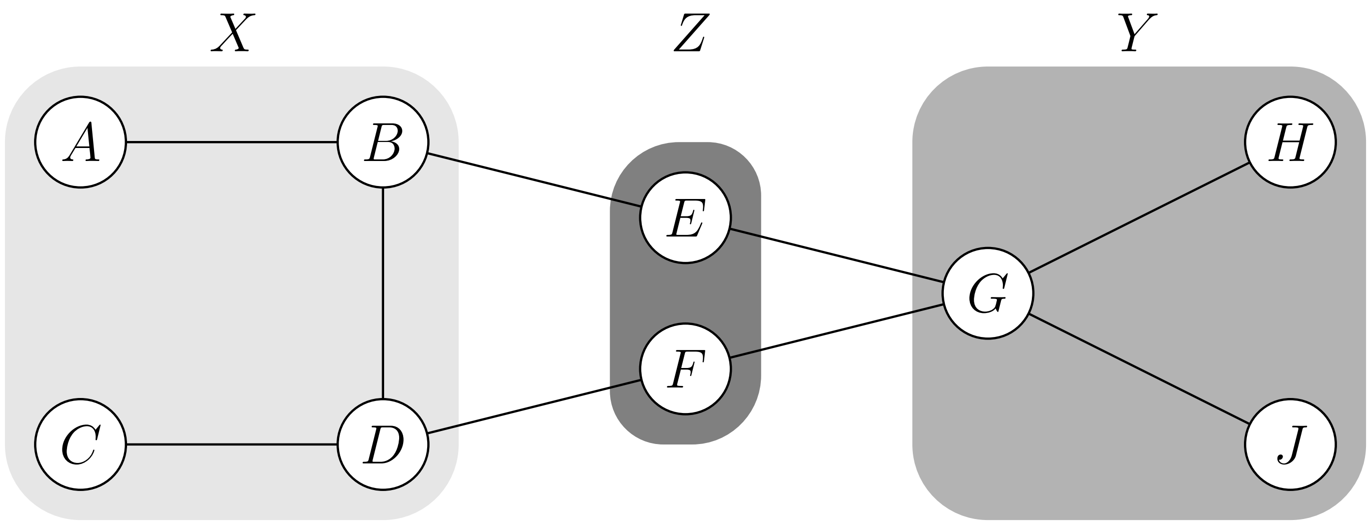
The **Moral Graph** of $\mathcal{G}$ is the undirected graph that is obtained by

1. connecting nodes that share a common child with an arbitrarily directed edge and,
2. converting all directed edges into undirected ones by dropping the arrow heads.



Moral graph of ancestral graph induced by the set $\{E, F, G\}$.

u-Separation



Let $\mathcal{G} = (V, E)$ be an undirected graph and $X, Y, Z \subseteq V$ three disjoint subsets of nodes. We agree on the following separation criteria:

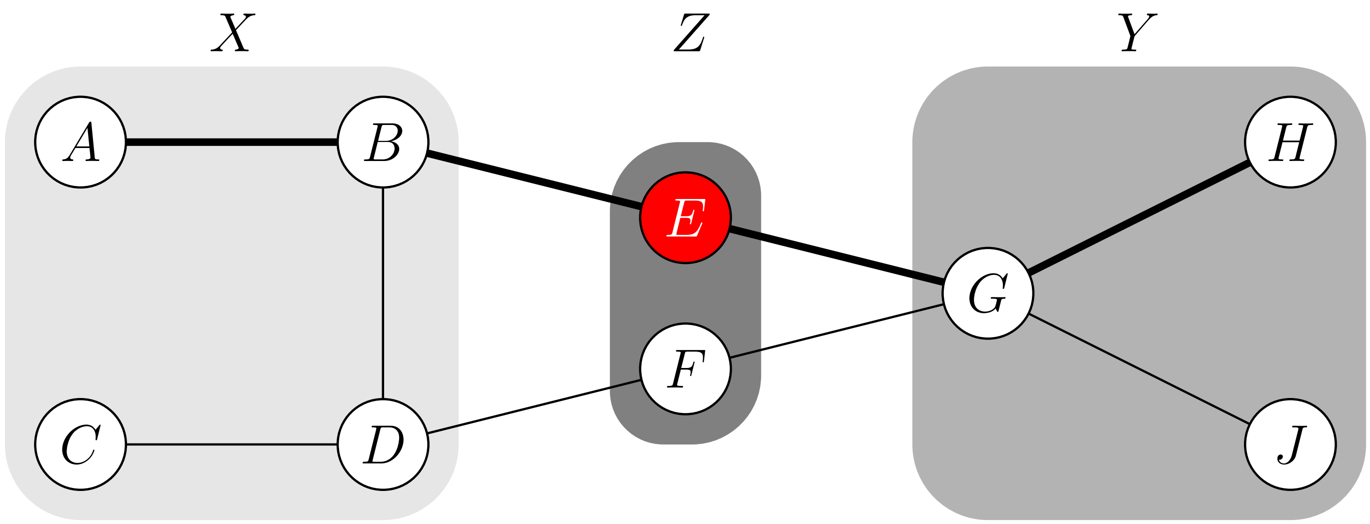
1. Z u-separates X from Y — written as

$$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z,$$

   if every possible path from a node in X to a node in Y is blocked.

2. A path is blocked if it contains one (or more) **blocking nodes**.
3. A node is a blocking node if it lies in Z.

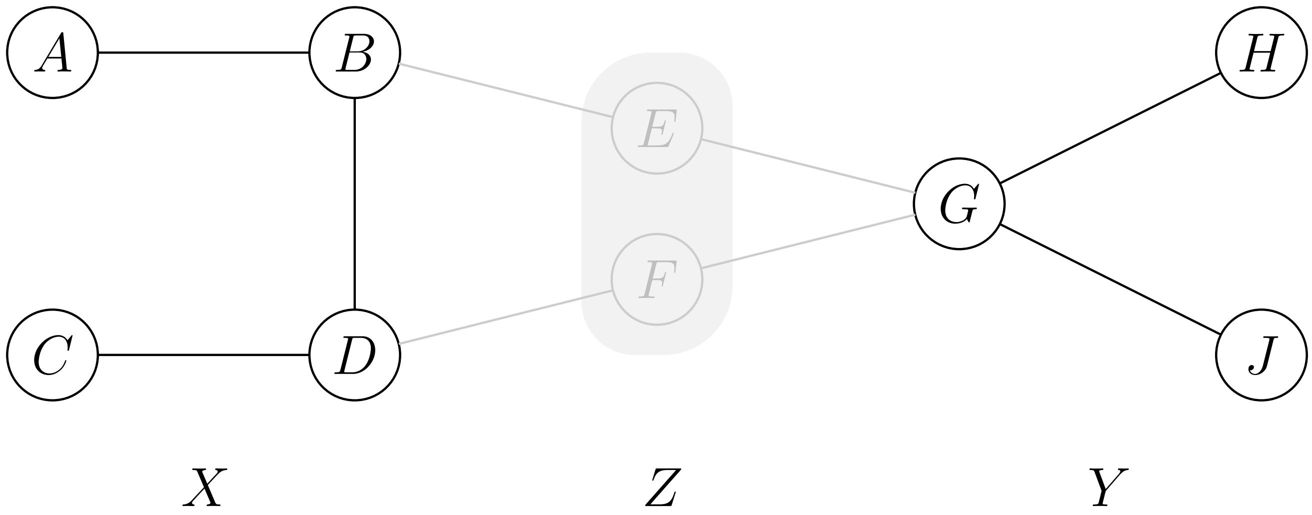
u-Separation



E. g. path $A - B - E - G - H$ is blocked by $E \in Z$. It can be easily verified, that every path from X to Y is blocked by Z. Hence we have:

$$\{A, B, C, D\} \perp\!\!\!\perp_{\mathcal{G}} \{G, H, J\} \mid \{E, F\}$$

u-Separation



Another way to check for u-separation: Remove the nodes in Z from the graph (and all the edges adjacent to these nodes). X and Y are u-separated by Z if the remaining graph is disconnected with X and Y in separate subgraphs.

d-Separation

Now: Separation criterion for directed graphs.

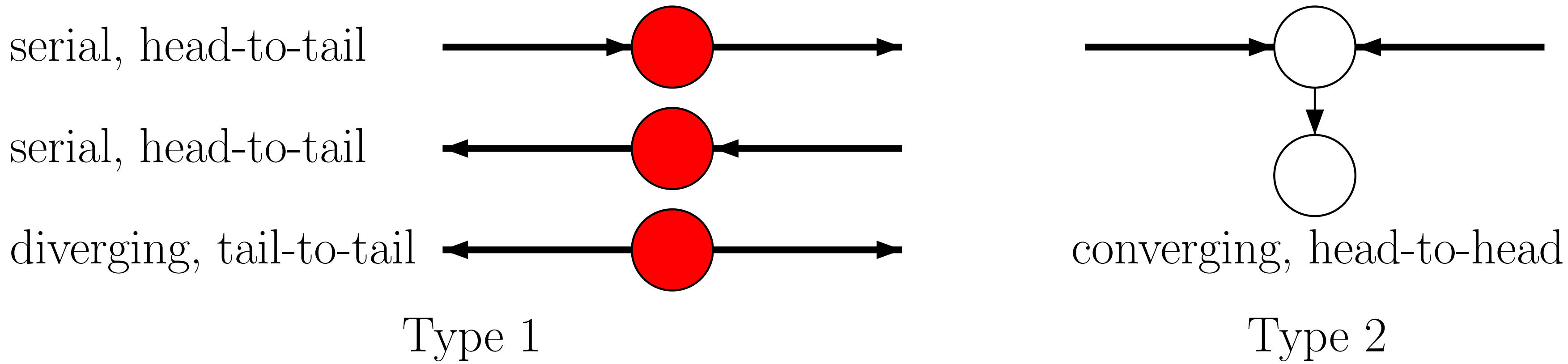
We use the same principles as for u-separation. Two modifications are necessary:

- Directed paths may lead also in reverse to the arrows.
- The blocking node condition is more sophisticated.

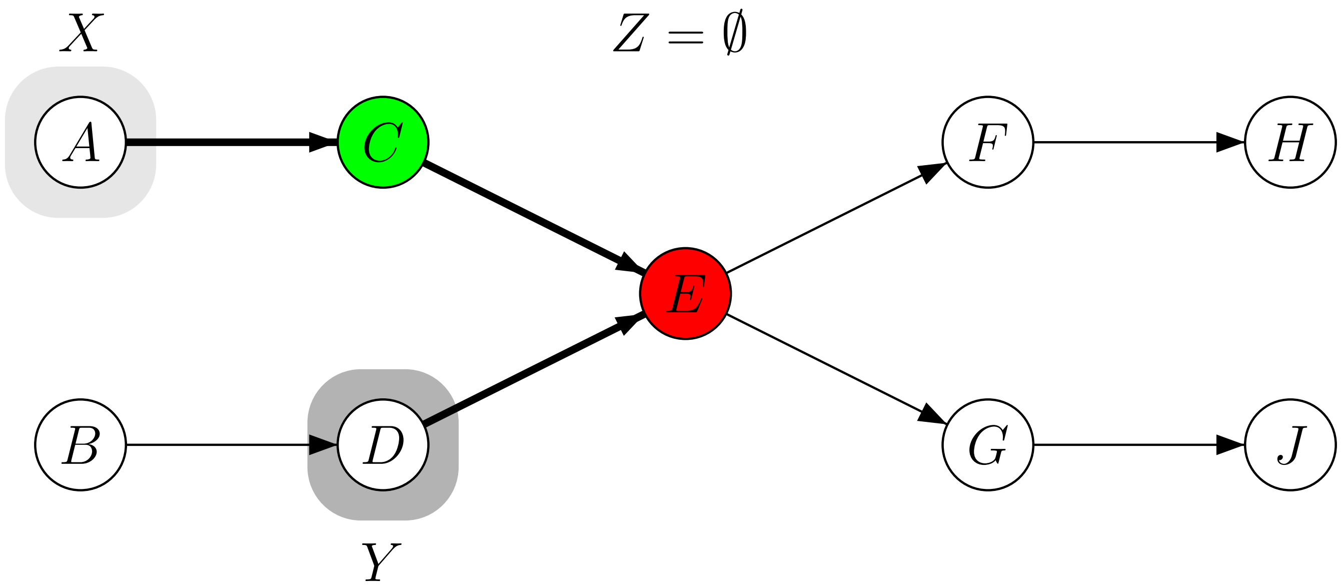
Blocking Node (in a directed path)

A node A is blocked if its edge directions **along the path**

- are of type 1 and $A \in Z$, or
- are of type 2 and neither A nor one of its descendants is in Z.



d-Separation



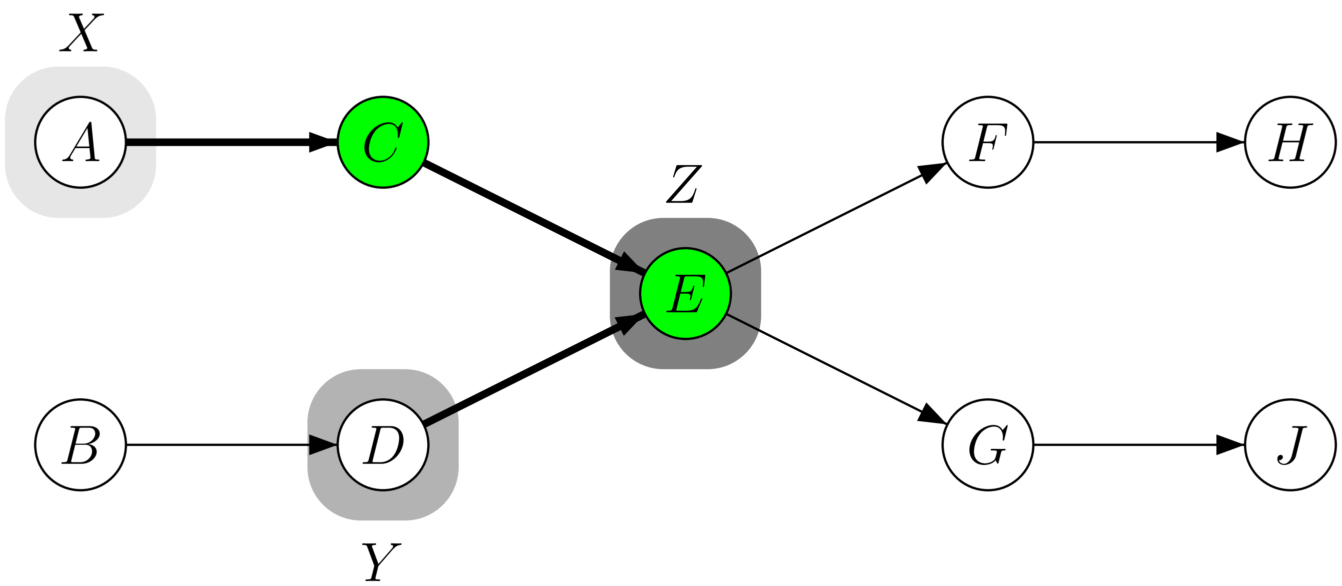
Checking path $A \rightarrow C \rightarrow E \leftarrow D$:

- C is **serial** and not in Z: non-blocking
- E is **converging** and not in Z, neither is F, G, H or J: **blocking**

$\Rightarrow$ Path is blocked

$$A \perp\!\!\!\perp D \mid \emptyset$$

d-Separation



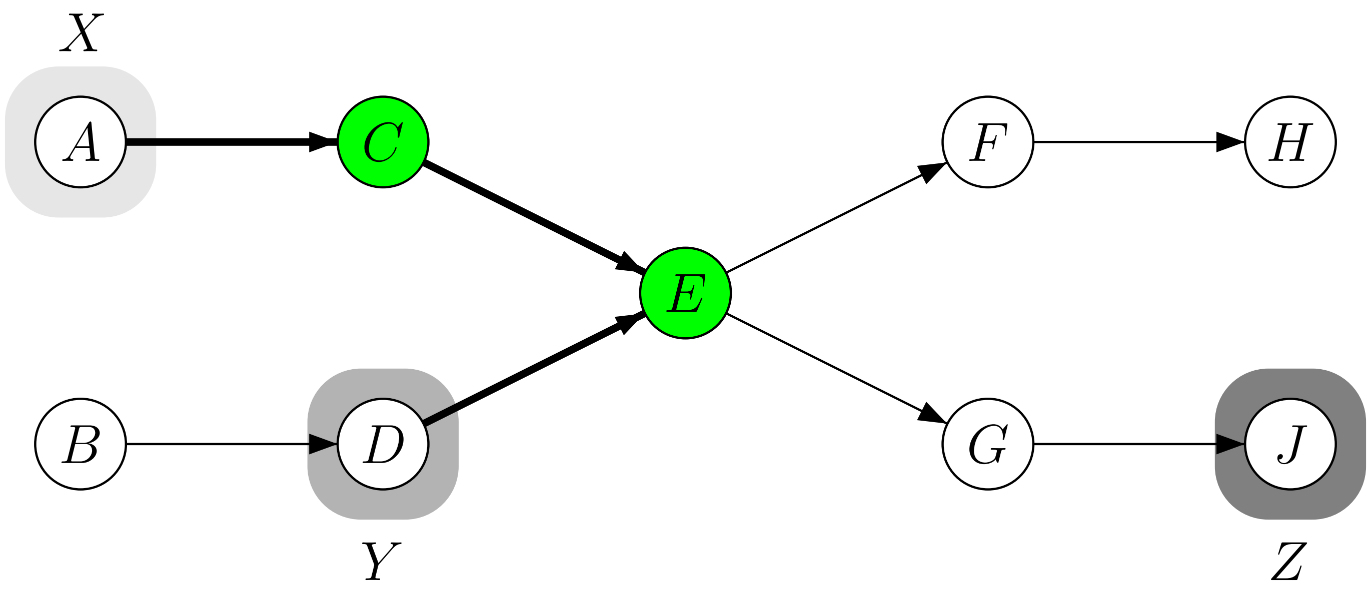
Checking path $A \to C \to E \leftarrow D$:

- C is **serial** and not in Z: non-blocking
- E is **converging** and in Z: non-blocking

$\Rightarrow$ Path is not blocked

$$A \not\perp\!\!\!\perp D \mid E$$

d-Separation



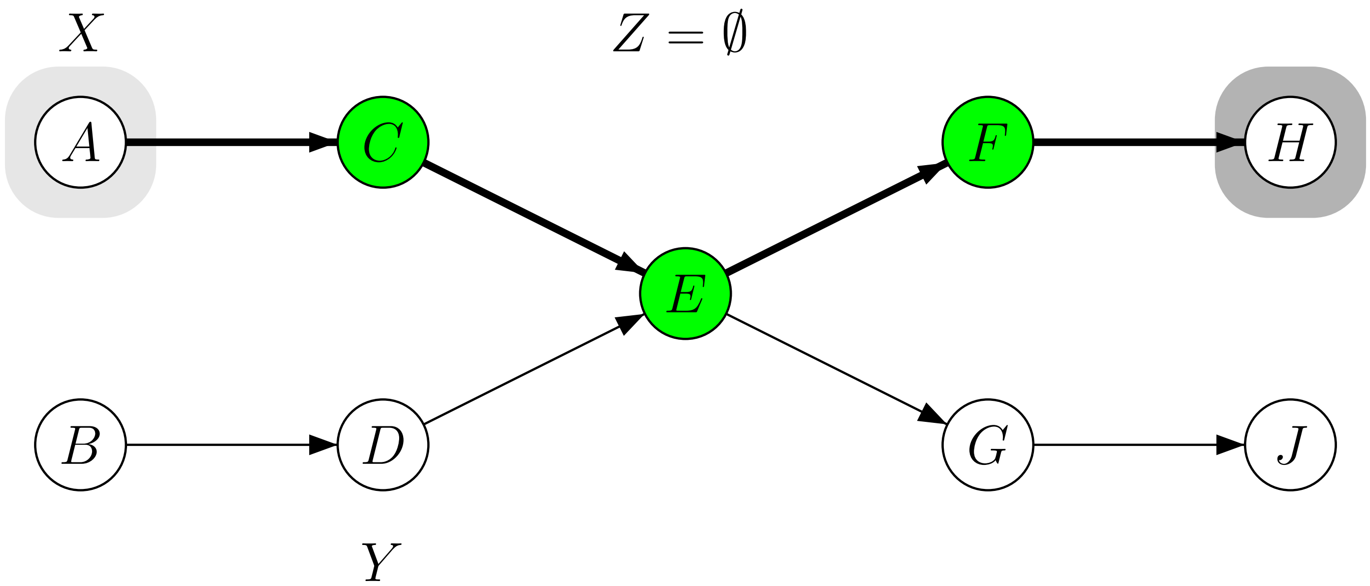
Checking path $A \rightarrow C \rightarrow E \leftarrow D$:

- C is **serial** and not in Z: non-blocking
- E is **converging** and not in Z but one of its descendants (J) is in Z: non-blocking

$\Rightarrow$ Path is not blocked

$$A \not\perp\!\!\!\perp D \mid J$$

d-Separation



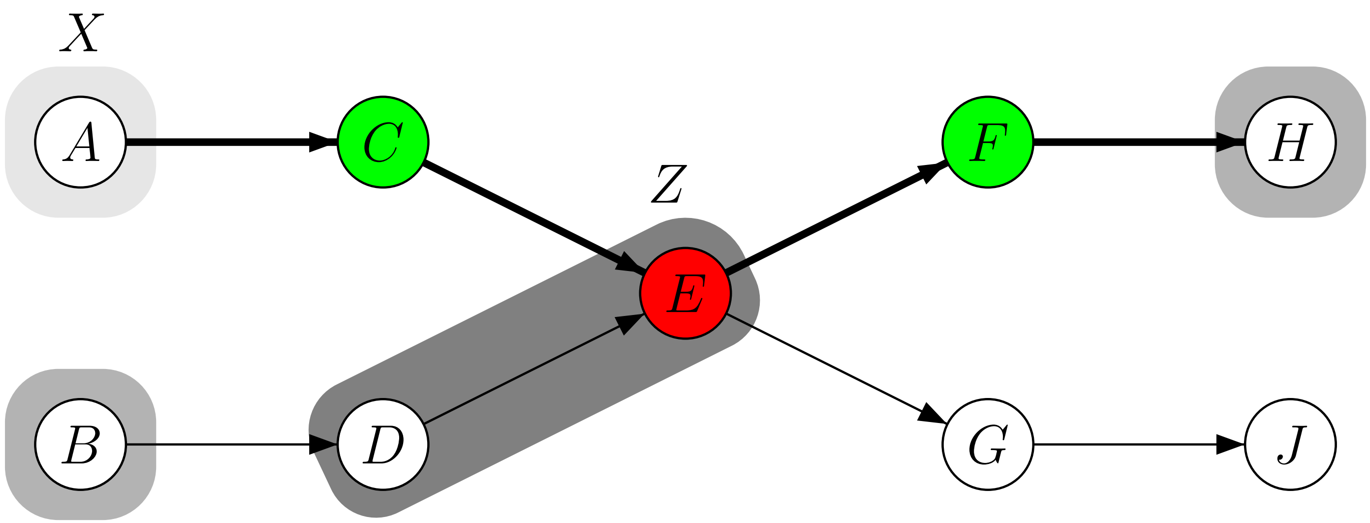
Checking path $A \to C \to E \to F \to H$:

- C is **serial** and not in Z: non-blocking
- E is **serial** and not in Z: non-blocking
- F is **serial** and not in Z: non-blocking

$\Rightarrow$ Path is not blocked

$$A \not\perp\!\!\!\perp H \mid \emptyset$$

d-Separation



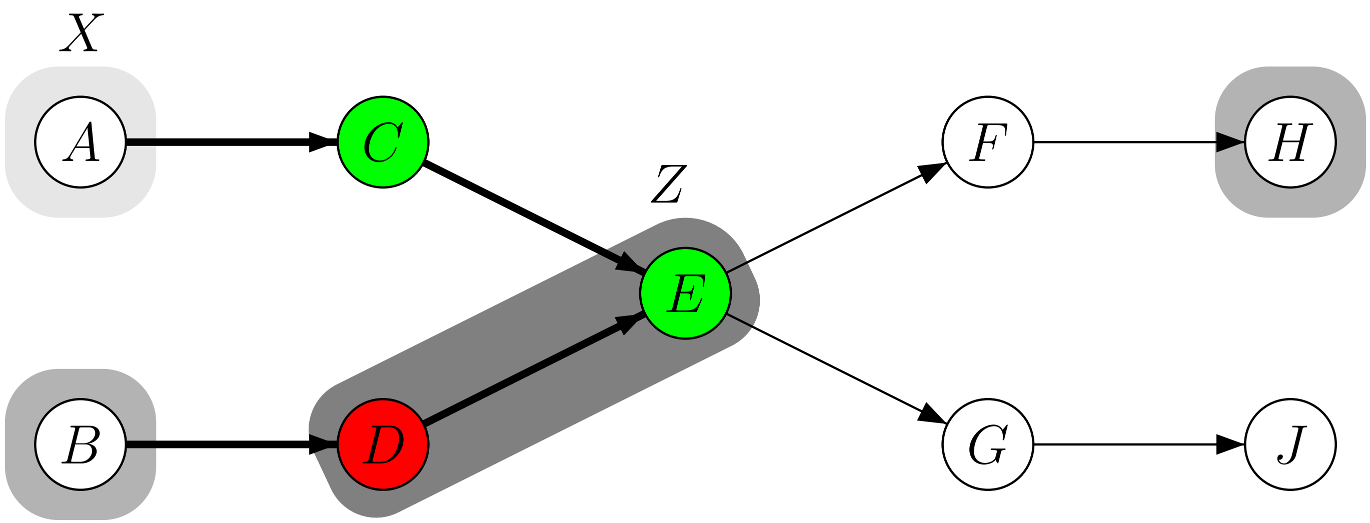
$Y = \{B, H\}$

Checking path $A \rightarrow C \rightarrow E \rightarrow F \rightarrow H$:

- C is **serial** and not in Z: non-blocking
- E is **serial** and in Z: **blocking**
- F is **serial** and not in Z: non-blocking

$\Rightarrow$ Path is blocked

d-Separation



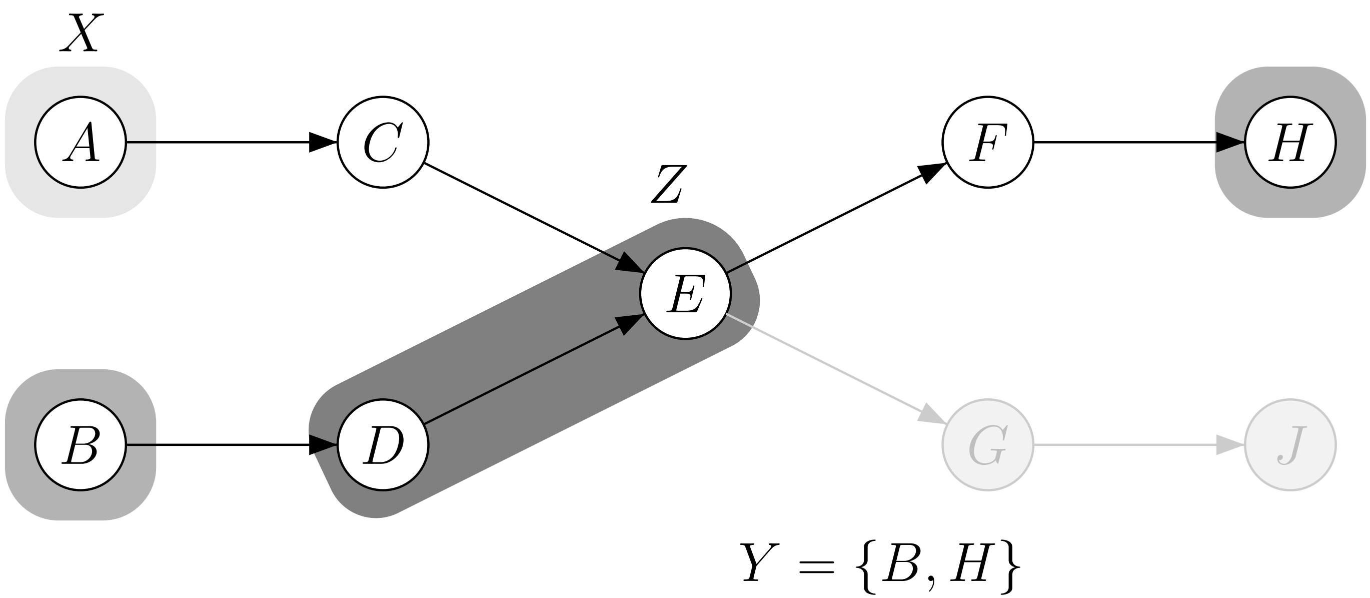
$Y = \{B, H\}$

Checking path $A \to C \to E \leftarrow D \to B$:

- C is **serial** and not in Z: non-blocking
- E is **converging** and in Z: non-blocking
- D is **serial** and in Z: **blocking**

$\Rightarrow$ Path is blocked

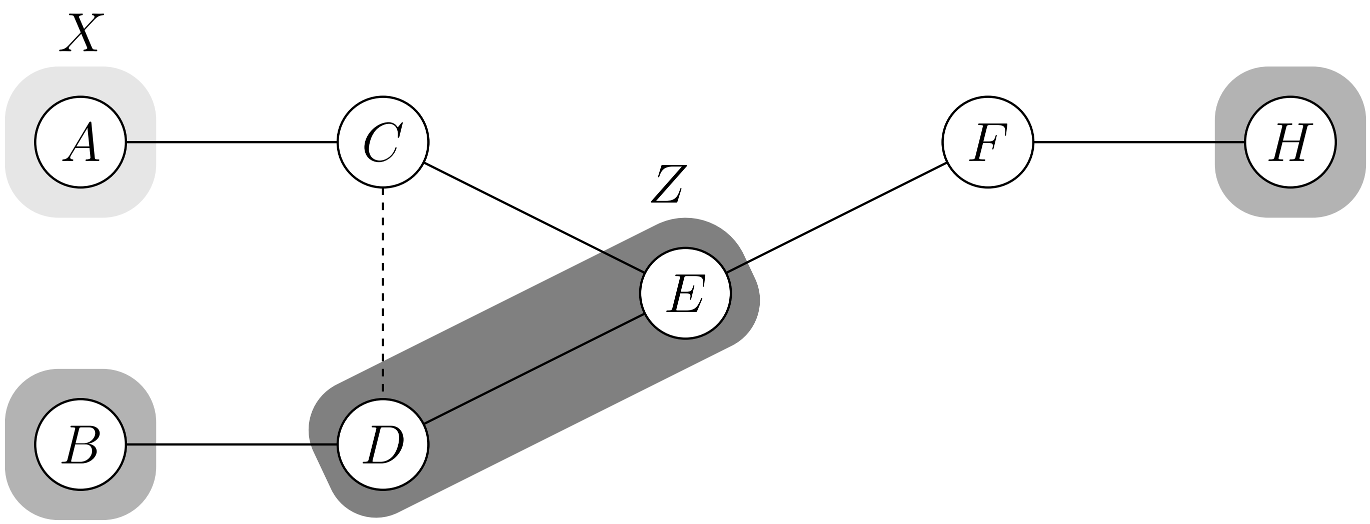
$$A \perp\!\!\!\perp H, B \mid D, E$$



Steps

- Create the minimal ancestral subgraph induced by $X \cup Y \cup Z$.

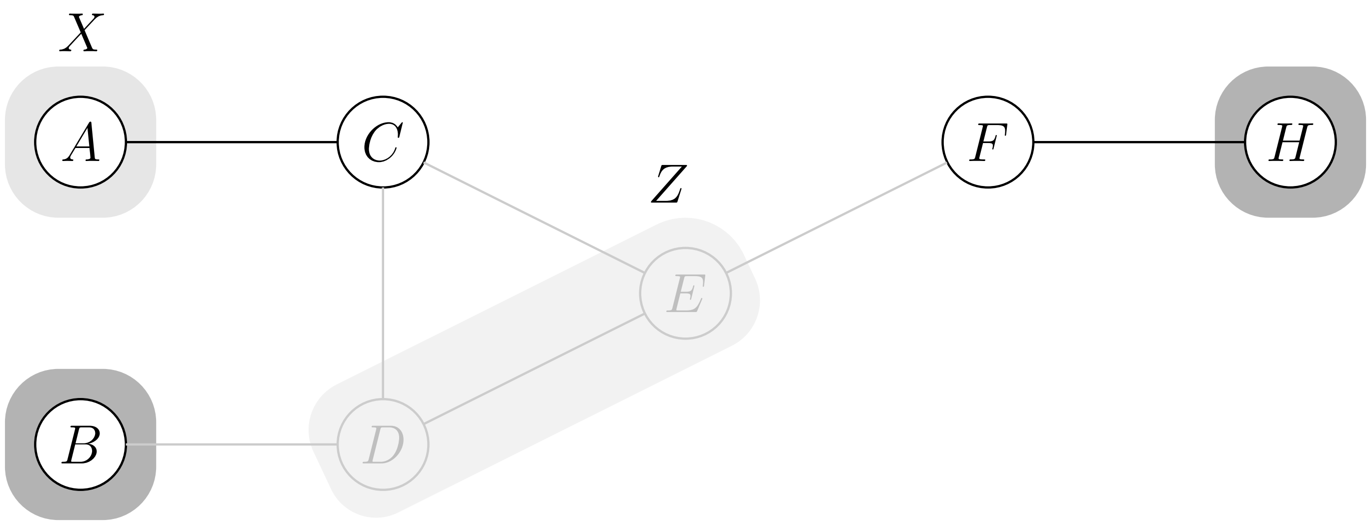
d-Separation: Alternative Way for Checking



$Y = \{B, H\}$

Steps

- Create the minimal ancestral subgraph induced by $X \cup Y \cup Z$.
- Moralize that subgraph.



$$Y = \{B, H\}$$

Steps:

- Create the minimal ancestral subgraph induced by $X \cup Y \cup Z$.
- Moralize that subgraph.
- Check for u-Separation in that undirected graph.

$$A \perp\!\!\!\perp H, B \mid D, E$$