Probability Foundations

Reminder: Probability Theory

- **Goal**: Make statements and/or predictions about results of physical processes.
- Even processes that seem to be simple at first sight may reveal considerable difficulties when trying to predict.
- Describing real-world physical processes always calls for a simplifying mathematical model.
- Although everybody will have some intuitive notion about probability, we have to formally define the underlying mathematical structure.
- Randomness or chance enters as the incapability of precisely modelling a process or the inability of measuring the initial conditions.
    - *Example*: Predicting the trajectory of a billard ball over more than 9 banks requires more detailed measurement of the initial conditions (ball location, applied momentum etc.) than physically possible according to Heisenberg's uncertainty principle.

Formal Approach on the Model Side

- We conduct an experiment that has a set Ω of possible outcomes.
  E. g.:
    - Rolling a die ($\Omega = \{1, 2, 3, 4, 5, 6\}$)
    - Arrivals of phone calls ($\Omega = \mathbb{N}_0$)
    - Bread roll weights ($\Omega = \mathbb{R}_+$)

- Such an outcome is called an **elementary event**.

- All possible elementary events are called the **frame of discernment** Ω (or sometimes **universe of discourse**).

- The set representation stresses the following facts:
    - All possible outcomes are covered by the elements of Ω.
      (**collectively exhaustive**).
    - Every possible outcome is represented by exactly one element of Ω.
      (**mutual disjoint**).

Events

- Often, we are interested in *higher-level* events (e. g. casting an odd number, arrival of at least 5 phone calls or purchasing a bread roll heavier than 80 grams)
- Any subset $A \subseteq \Omega$ is called an **event** which **occurs**, if the outcome $\omega_0 \in \Omega$ of the random experiment lies in A:

$$\text{Event } A \subseteq \Omega \ \text{ occurs} \quad \Leftrightarrow \quad \bigvee_{\omega \in A} (\omega = \omega_0) \ = \ \mathsf{true} \quad \Leftrightarrow \quad \omega_0 \in A$$

- Since events are sets, we can define for two events A and B:
  - $A \cup B$ occurs if A or B occurs; $A \cap B$ occurs if A and B occurs.
  - $\overline{A}$ occurs if A does not occur (i. e., if $\Omega \backslash A$ occurs).
  - A and B are *mutually exclusive*, iff $A \cap B = \emptyset$.

Event Algebra

- A family of sets $\mathcal{E} = \{E_1, \ldots, E_n\}$ is called an **event algebra**, if the following conditions hold:
  - The **certain event** Ω lies in $\mathcal{E}$.
  - If $E \in \mathcal{E}$, then $\overline{E} = \Omega \backslash E \in \mathcal{E}$.
  - If E_1 and E_2 lie in $\mathcal{E}$, then $E_1 \cup E_2 \in \mathcal{E}$ and $E_1 \cap E_2 \in \mathcal{E}$.
- If Ω is uncountable, we require the additional property:

  For a series of events $E_i \in \mathcal{E}, i \in \mathbb{N}$, the events $\bigcup_{i=1}^{\infty} E_i$ and $\bigcap_{i=1}^{\infty} E_i$ are also in $\mathcal{E}$.

  $\mathcal{E}$ is then called a **σ-algebra**.

Side remarks:

- Smallest event algebra: $\mathcal{E} = \{\emptyset, \Omega\}$
- Largest event algebra (for finite or countable Ω): $\mathcal{E} = 2^\Omega = \{A \subseteq \Omega \mid \mathsf{true}\}$

Probability Function

- Given an event algebra $\mathcal{E}$, we would like to assign every event $E \in \mathcal{E}$ its probability with a **probability function** $P: \mathcal{E} \to [0,1]$.
- We require P to satisfy the so-called **Kolmogorov Axioms**:
  - $\forall E \in \mathcal{E}: \; 0 \leq P(E) \leq 1$
  - $P(\Omega) = 1$
  - If $E_1, E_2 \in \mathcal{E}$ are mutually exclusive, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.
- From these axioms one can conclude the following (incomplete) list of properties:
  - $\forall E \in \mathcal{E}: \; P(\overline{E}) = 1 - P(E)$
  - $P(\emptyset) = 0$
  - For pairwise disjoint events $E_1, E_2, \ldots \in \mathcal{E}$ holds:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

    Note that for $|\Omega| < \infty$ the union and sum are finite also.

Elementary Probabilities and Densities

Question 1: How to calculate P?

Question 2: Are there "default" event algebras?

- Idea for question 1: We have to find a way of distributing (thus the notion *distribution*) the unit mass of probability over all elements $\omega \in \Omega$.
  - If Ω is finite or countable a **probability mass function** p is used:

$$p: \ \Omega \to [0,1] \quad \text{and} \quad \sum_{\omega \in \Omega} p(\omega) = 1$$

  - If Ω is uncountable (i. e., continuous) a **probability density function** f is used:

$$f: \ \Omega \to \mathbb{R} \quad \text{and} \quad \int_{\Omega} f(\omega) \, \mathrm{d}\omega = 1$$

"Default" Event Algebras

- Idea for question 2 ("default" event algebras) we have to distinguish again between the cardinalities of Ω:
  - Ω finite or countable: $\mathcal{E} = 2^\Omega$
  - Ω uncountable, e. g. $\Omega = \mathbb{R}$: $\mathcal{E} = \mathcal{B}(\mathbb{R})$
- $\mathcal{B}(\mathbb{R})$ is the **Borel Algebra**, i. e., the smallest σ-algebra that contains all closed intervals $[a, b] \subset \mathbb{R}$ with $a < b$.
- $\mathcal{B}(\mathbb{R})$ also contains all open intervals and single-item sets.
- It is sufficient to note here, that all intervals are contained

$$\{[a,b]\,,\,]a,b]\,,\,]a,b[\,,\,[a,b[\, \subset \mathbb{R} \mid a < b\} \subset \mathcal{B}(\mathbb{R})$$

  because the event of a bread roll having a weight between 80 g and 90 g is represented by the interval $[80, 90]$.

Probability Spaces

- For a sample space A, an event algebra B (over A) and a probability function C, we call the triple

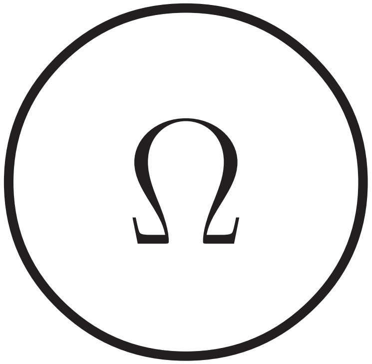
$$(A, B, C)$$

a **probability space**.

Real World

$(\Xi, \mathcal{X}, Q)$

Model



$(\Omega, \mathcal{E}, P)$

Reminder: Preimage of a Function

- Let $f: D \to M$ be a function that assigns to every value of D a value in M.
- For every value of $y \in M$ we can ask which values of $x \in D$ are mapped to y:

$$D \supseteq \{x \in D \mid f(x) = y\} \stackrel{\text{Def}}{=} f^{-1}(y)$$

- $f^{-1}(y)$ is called the **preimage** of y under f, denoted also as $\{f = y\}$.
- The notion can be generalized from $y \in M$ to sets $B \subseteq M$:

$$D \supseteq \{x \in D \mid f(x) \in B\} \stackrel{\text{Def}}{=} f^{-1}(B)$$

- If f is bijective then $\forall y \in M : \left| f^{-1}(y) \right| = 1$.
- Examples:
  - $\sin^{-1}(0) = \{k \cdot \pi \mid k \in \mathbb{Z}\}$
  - $\exp^{-1}(1) = \{0\}$
  - $\text{sgn}^{-1}(1) = (0, +\infty) \subset \mathbb{R}$

Random Variable

We still need a means of mapping real-world outcomes in Ξ to our space Ω.

- A function $X: D \to M$ is called a **random variable** iff the preimage of any value of M is an event (in some probability space).

- If X maps Ξ onto Ω, we define

$$P_X(X \in A) \;=\; Q(\{\xi \in \Xi \mid X(\xi) \in A\}).$$

- X may also map from Ω to another domain: $X: \Omega \to \mathrm{dom}(X)$.
  We then define:

$$P_X(X \in A) \;=\; P(\{\omega \in \Omega \mid X(\omega) \in A\}).$$

- If X is numeric, we call $F(x)$ with

$$F(x) \;=\; P(X \leq x)$$

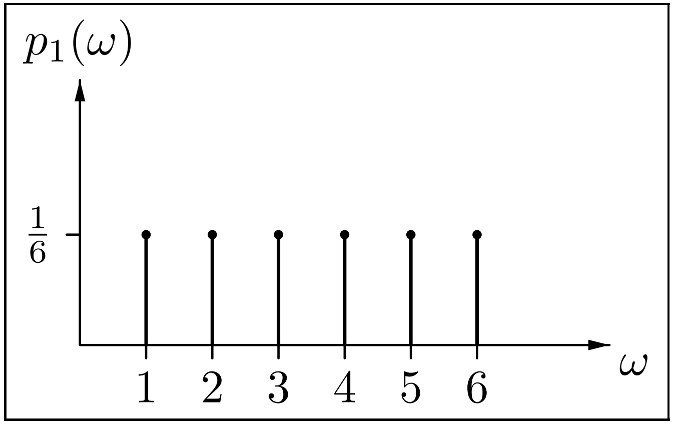
  the **distribution function** of X.

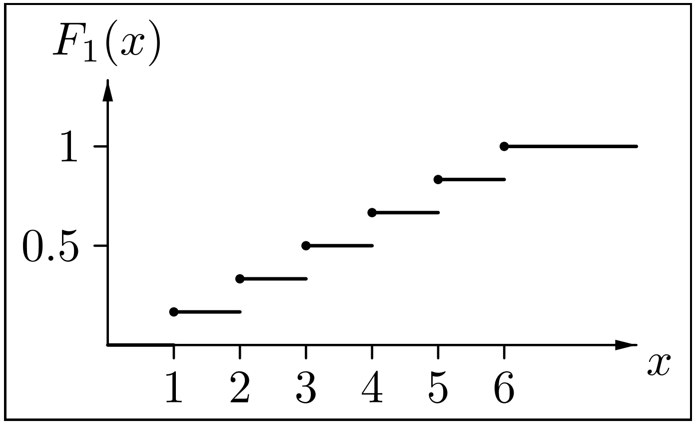
Example: Rolling a Die

$\Omega = \{1, 2, 3, 4, 5, 6\} \quad X = \text{id}$

$p_1(\omega) = \frac{1}{6}$

$F_1(x) = P(X \le x)$



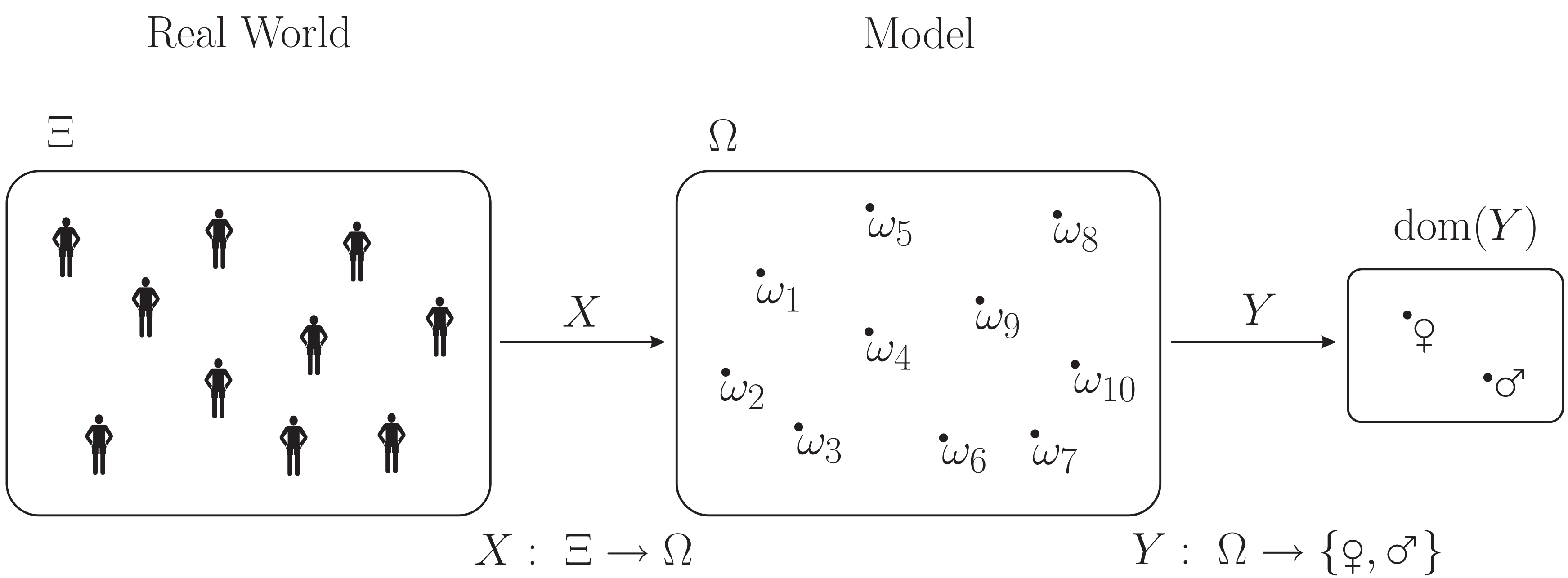


$$\sum_{\omega \in \Omega} p_1(\omega) = \sum_{i=1}^{6} p_1(\omega_i) = \sum_{i=1}^{6} \frac{1}{6} = 1$$

$$P(X \le x) = \sum_{x' \le x} P(X = x')$$

$$P(a < X \le b) = F_1(b) - F_1(a)$$

$$P(X = x) = P(\{X = x\}) = P(X^{-1}(x)) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$



$$Q\Big(\{\xi \in \Xi \mid X(\xi) \in Y^{-1}(♀)\}\Big) \quad = \quad P\Big(\{\omega \in \Omega \mid Y(\omega) = ♀\}\Big) \quad = \quad P\Big(Y = ♀\Big) \quad = \quad P\Big(♀\Big)$$