2. Elementary Fuzzy Sets

Def. 2.1

Let $X \neq \emptyset$ be a set.

a) $2^X := \{A | A \subseteq X\}$ power set of X

b) $A \in 2^X$, $I_A : X \to \{0,1\}$, Indicator function

c) $CH(X) := \{I_A | A \in 2^X\}$ Set of indicator functions.

Theorem 2.2

a) $(2^X, \cap, \cup, C)$ Boolean algebra

b) $\phi: 2^X \to CH(X)$, $\phi(A) := I_A$ bijection

Theorem 2.3

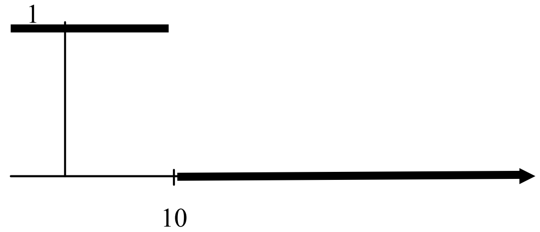
$(CH(X), \wedge, \vee, \neg)$ is Boolean algebra, where

$\mu_1 \wedge \mu_2 := min\ \{\mu_1, \mu_2\}$, $\mu_1 \vee \mu_2 := max\ \{\mu_1, \mu_2\}$, $\overline{\mu} = 1 - \mu$

Example 2.4

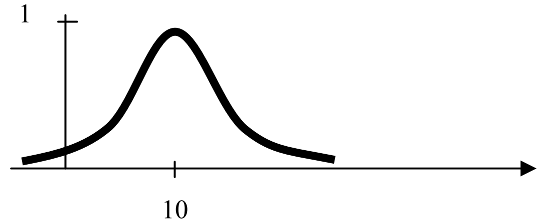
ling. description — model

all numbers smaller than 10 → objective



Indicator funktion

all numbers almost equal to 10 → subjective



Membership function of "fuzzy set"

Def.: 2.5

a) A fuzzy set μ of $X \neq \emptyset$ is a function from the reference set X to the unit interval, i.e. $\mu: X \rightarrow [0,1]$.

b) $F(X) := \{\mu \ \mu: X \rightarrow [0,1]\}$

Example 2.6

"Subjective definition of a fuzzy Set"

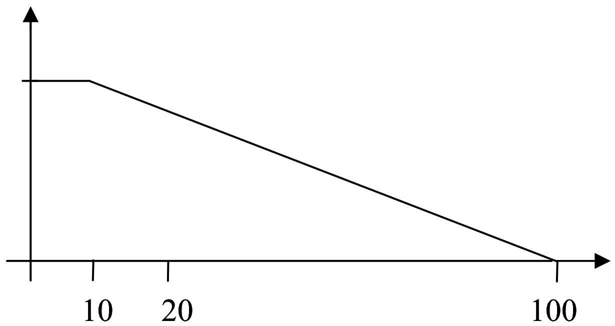
X:= set of Magdeburg residents between 10 and 100 years of age

Y={1,…100}

j(Y) number of residents aged y who consider themselves "young"

n(y) total number of residents aged y

$$\mu: Y \to [0,1],\ \mu(y) = \begin{cases} \dfrac{j(y)}{n(y)} & \text{if } y > 10 \\ 1 & \text{if } y \leq 10 \end{cases}$$



Def. 2.7

a) We define on F(X) the following operations:

$(\mu \wedge \mu')(x) := \min\{\mu(x), \mu'(x)\}$ intersection (and)

$(\mu \vee \mu')(x) := \max\{\mu(x), \mu'(x)\}$ junction (or)

$\neg\mu(x) = 1-\mu(x)$ complement

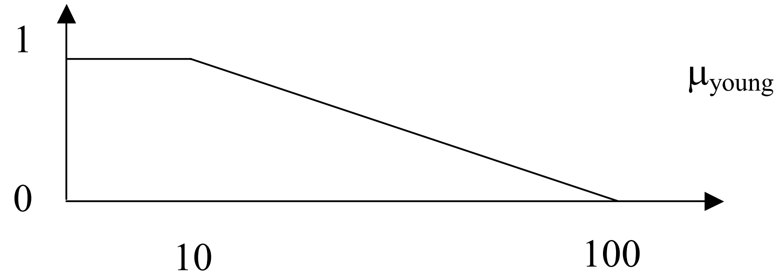
b) μ is subset of μ' $\Leftrightarrow$ $\mu \leq \mu'$.

Theorem 2.8

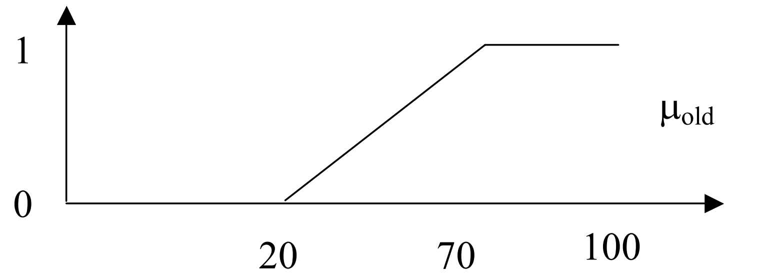
$(F(X),\wedge,\vee,\neg)$ is a complete distributive lattice, however no boolean algebra.

Example 2.9

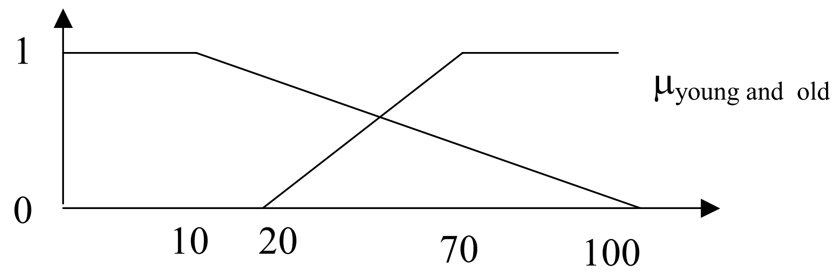
young:



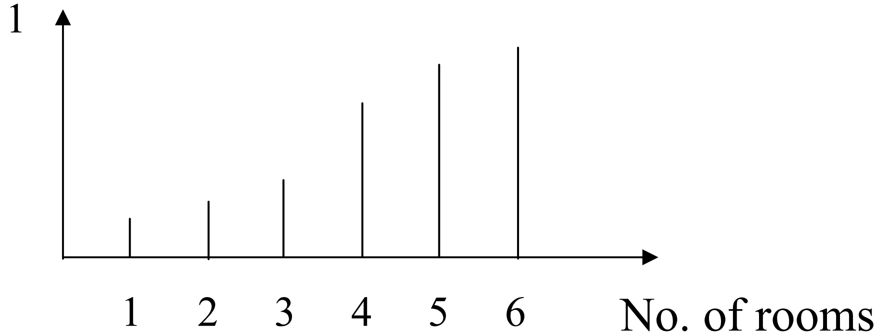
old:



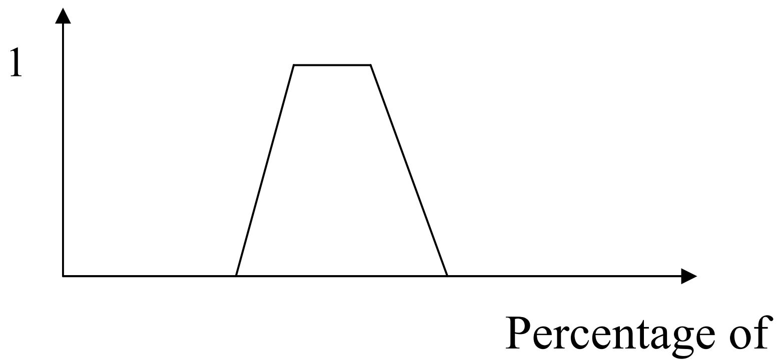
young and old:



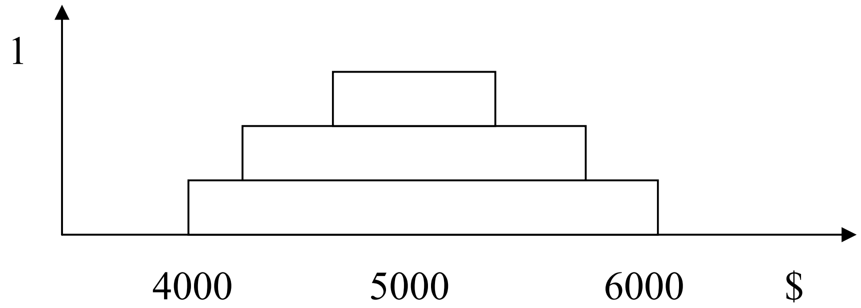
Example 2.10 A large appartment



Example 2.11 „cloudy“



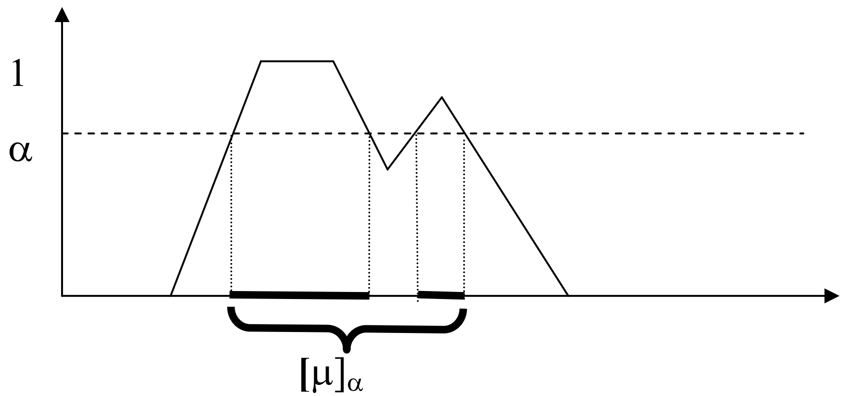
Example 2.12 nearly 5000 $



Def 2.13

Let μ ∈ F(X) and α ∈ [0,1]. Then the set $[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called the α-cut or α-level set of μ.

Example 2.14 α-cut of a fuzzy set



Theorem 2.15

Let $\mu \in F(X)$, $\alpha \in [0,1]$, $\beta \in (0,1]$.

a) $[\mu]_0 = X$

b) $\alpha < \beta \Rightarrow [\mu]_\alpha \supseteq [\mu]_\beta$

c) $\bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$ for all $\beta \in [0,1]$

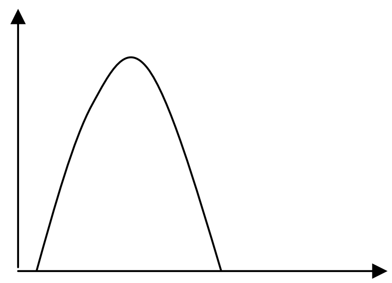
Theorem 2.16 (Negoita)
Let $\mu \in F(X)$. Then $\mu(t) = \sup\limits_{\alpha \in [0,1]} \{\alpha \wedge I_{\mu_\alpha}(t)\}$ for all $\mu \in F(X)$.

Geometrically speaking, a fuzzy set can be obtained as the upper envelope of its α-cuts, if we draw the α-cuts parallel to the horizontal axis in the height α.

Def 2.17
Let X be a vector space. A fuzzy set μ(X) is fuzzy convex if all α-cuts are convex sets.

Theorem 2.18
μ is fuzzy convex $\Leftrightarrow \underset{x_1, x_2 \in X}{\forall} \; \underset{\lambda \in [0,1]}{\forall} : \mu(\lambda x_1 + (1-\lambda) x_2) \geq \mu(x_1) \wedge \mu(x_2)$

Example 2.19

Def. 2.20

$\mu: X \to [0,1]$

$\operatorname{supp}(\mu) := \{x \in X \mid \mu(x) > 0\}$ is the carrier of μ.

Def. 2.21

$\mu: \Re \to [0,1]$

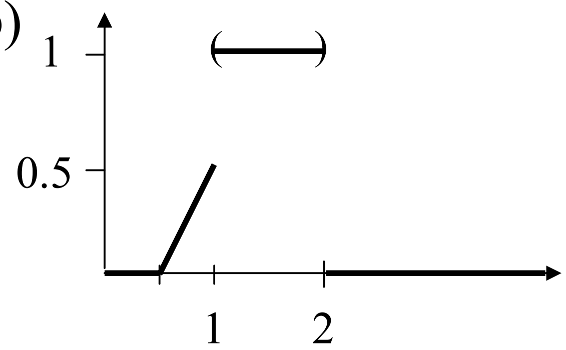
a) μ is a normal fuzzy set $\Leftrightarrow \underset{x \in \Re}{\exists} : \mu(x) = 1$

b) μ is a fuzzy number $\Leftrightarrow \underset{\alpha \in (0,1]}{\forall} : [\mu]_\alpha$ is limited, closed, and convex and μ is normal.

Example 2.22

a) approximately (x_0) is often described by a parametrized class of membership functions, e.g.
$\mu_1(x)=\max\{0,1-c_1|x-x_0|\}$, $c_1>0$
$\mu_2(x)=\exp\{-c_2,|x-x_0|^p\}$, $c_2>0$, $p\geq 1$

b)



$$[\mu]_\alpha = \begin{cases} (1,2) & \text{if } \alpha > 0.5 \\ [0.5+\alpha, 2) & \text{if } 0 < \alpha \le 0.5 \\ \Re & \text{if } \alpha = 0 \end{cases}$$

c) From a mathematical point of view upper semi continous functions are most convenient.

d) In many applications (such as fuzzy control) the function class of functions and its exact parameters have only limited influence on the results, because only the local monotonicity of functions is needed. In other applications (such as diagnosis) more precise membership degrees are needed.