6. Fuzzy Relations

Example 6.1

$\{(x_1, x_2) \in X^2 | \, x_1 = x_2\}$ equality

$\mu: X \times X \rightarrow [0,1]$, for all $x \in X$: $\mu(x,x)=1$, fuzzy equality

Def. 6.2

A fuzzy relation *R* on X×Y is defined by a membership function
$\mu_R : X \times Y \rightarrow [0,1]$.

Example 6.3

$S=\{s_1, s_2, s_3, s_4\}$, $H=\{h_1, h_2, h_3\}$

Crisp Relation

	h_1	h_2	h_3
s_1	1	0	0
s_2	0	1	0
s_3	0	1	1
s_4	0	0	1

Fuzzy Relation:

		h_1	h_2	h_3
	s_1	1	0.2	0
μ_R	s_2	0.1	1	0
	s_3	0	1	1
	s_4	0	0.3	1

Def 6.4

If A is a fuzzy set on X and B is a fuzzy set on Y, then we define the fuzzy cartesian product $C=A\otimes B$ by:

$\mu_C: X\times Y\rightarrow[0,1]$, where $\mu_C(a,b)=\min(\mu_A(a), \mu_B(b))$.

Remark 6.5

a) Every fuzzy cartesian product is a fuzzy relation.
b) Not any fuzzy relation R on $X\times Y$ is the cartesian product of 2 fuzzy sets A on X and B on Y. There are fuzzy sets A and B with $R\subseteq A\otimes B$
c) Every method defined for fuzzy sets can be studied for fuzzy relations as well. (Intersection, operator definition etc.)

Definition 6.6

a) $R \subseteq X \times Y$, $S \subseteq Y \times Z$:
$R \bullet S \subseteq X \times Z$ Concatenation of R and S,
$(x,z) \in R \bullet S \Leftrightarrow \exists\, y \in Y: \{(x,y) \in R \text{ and } (y,z) \in S\}$

b) $R \subseteq X \times Y : R^{-1} \subseteq Y \times X$ inverse of R,
$(x,y) \in R \Leftrightarrow (y,x) \in R^{-1}$

c) Let R be a fuzzy relation on X×Y, S a fuzzy relation on Y×Z, then the concatenation of R and S R•S is a fuzzy relation on X×Z, defined by:
$\mu_{R \bullet S}(x,z) = \sup_{y \in Y}\{\min(\mu_R(x,y), \mu_S(y,z)\}$

d) $\mu_{R^{-1}} : Y \times X \rightarrow [0,1]$, $\mu_{R^{-1}}(y,x) = \mu_R(x,y)$

Example 6.7

$$\mu_R = \begin{pmatrix} 1 & 0,2 & 0 \\ 0,1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0,3 & 1 \end{pmatrix}, \ \mu_S = \begin{pmatrix} 1 & 0 \\ 0,9 & 1 \\ 0 & 0,9 \end{pmatrix} \qquad \mu_{R \bullet S} = \begin{pmatrix} 1 & 0,2 \\ 0,9 & 1 \\ 0,9 & 1 \\ 0,3 & 0,9 \end{pmatrix}$$

Theorem 6.8

a) $(\mu_{R\bullet S})_\alpha = (\mu_R)_\alpha \bullet (\mu_S)_\alpha \quad \alpha \geq 0$

b) $(R\bullet S)\bullet I = R\bullet(S\bullet I)$

c) $R\bullet(S\cup T) = (R\bullet S)\cup(R\bullet T)$

d) $R\bullet(S\cap T) \subseteq (R\bullet S)\cap(R\bullet T)$

e) $(R\bullet S)^{-1} = S^{-1}\bullet R^{-1}$

f) $(R\cup S)^{-1} = R^{-1}\cup S^{-1}$, $(R\cap S)^{-1} = R^{-1}\cap S^{-1}$

g) $(R^{-1})^{-1} = R$

h) $R\subseteq S \Rightarrow \{R\bullet T \subseteq S\bullet T,\ T\bullet R \subseteq T\bullet S\}$

Def. 6.9

Let R be a fuzzy relation on $X_1 \times X_2 \times X_3$.

$\pi_1 R$ is a projection from R on X_1, with

$$\mu_{\pi_1 R}(a) := \sup_{(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3 : \pi_1 R(x_1, x_2, x_3) = a} \{\mu_R(x_1, x_2, x_3)\}$$

$$\mu_{\pi_1 R}(a) := \sup_{x_2 \in X_2, x_3 \in X_3} \{\mu_R(a, x_2, x_3)\}$$

Def. 6.10

Let R be a fuzzy relation on $X_1 \times X_2$.

$\hat{\pi}_3 R$ is a cylindical extension from R on $X_1 \times X_2 \times X_3$, with

$$\mu_{\hat{\pi}_3 R}(x_1, x_2, x_3) := \mu_R(x_1, x_2)$$

Def. 6.11

A fuzzy relation $R \subseteq X_1 \times X_2 \times \ldots \times X_n$ is separable, if

$R = \pi_1 R \otimes \pi_2 R \otimes \ldots \otimes \pi_n R$ holds.

Example 6.12

Fuzzy Relation