Exercise Sheet 4

Exercise 13 Estimators

Let $X = (X_1, \ldots, X_n)$ be the random vector underlying a random sample of size n. We assume that the random variables X_i are independent and identically distributed according to the exponential distribution $f_X(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$, $x > 0$. We desire to estimate the parameter θ of this distribution. The most commonly used estimator for θ is $W_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$. Here, however, we consider the estimator $W_2 = n \cdot X_{\min} = n \min_{i=1}^{n} X_i$. Determine the probability density function of this estimator, that is, $f_{W_2}(w;\theta)$.
Hint: Recall the technical trick to consider the complementary event instead of the event itself, which we already used in exercise 1.

Exercise 14 Properties of Estimators

Show: the relative frequency r_A, with which an event A occurs in a given random sample of size n, is a consistent and unbiased estimator for the parameter $p = P(A)$ of a binomial distribution $b_X(x; p, n)$. (p is the probability, with which A occurs in a single instance of the random experiment — which is a Bernoulli experiment).
Hint: Consider the arithmetical mean of n independent random variables $Y_1, \ldots, Y_n$ for n Bernoulli experiments with

$$Y_i = \begin{cases} 1, & \text{if event } A \text{ occurs in the } i\text{-th trial,} \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 15 Unbiasedness of Estimators

a) Let W_1 and W_2 be two unbiased estimators for the unknown parameter θ. If we want $W = aW_1 + bW_2$ to be an unbiased estimator for θ as well, what conditions must hold for a and b?

b) Show: if we desire to estimate the parameter $\mu = E(X)$, then $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is always an unbiased estimator for μ.

Exercise 16 Efficiency of Estimators

In the lecture we considered $W_1 = \frac{n+1}{n} X_{\max} = \frac{n+1}{n} \max_{i=1}^{n} X_i$ as an unbiased estimator for the parameter θ of a uniform distribution on the interval $[0, \theta]$. As an alternative, one may use the (also unbiased) estimator $W_2 = (n+1)X_{\min} = (n+1)\min_{i=1}^{n} X_i$, which can be derived in a similar way. Which of these two estimators is more efficient?