Exercise Sheet 11

Exercise 39 Fuzzy Clustering

Consider the objective function of fuzzy clustering with a fuzzifier $w = 1$, that is,

$$J(\mathbf{X}, \mathbf{B}, \mathbf{U}) = \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}\, d^2(\beta_i, \vec{x}_j),$$

which is to be minimized under the constraint

$$\forall j \in \{1, \dots, n\} : \quad \sum_{i=1}^{c} u_{ij} = 1.$$

Show that one obtains a hard/crisp assignment of the data points even if the membership degrees u_{ij} may come from the interval $[0, 1]$. That is, show that for the minimum of the objective function J it is $\forall i \in \{1, \dots, c\} : \forall j \in \{1, \dots, n\} : u_{ij} \in \{0, 1\}$.
(Hint: You may find it easier to consider the special case $c = 2$ (two clusters) and to examine the term for a single data point $\vec{x}_j$. Then generalize the result.)

Exercise 40 Agglomerative Clustering

Let the following one-dimensional data set be given:

$$2, 5, 11, 12, 17, 21, 32.$$

Process this data set with hierarchical agglomerative clustering using

a) the centroid method,
b) the single linkage methode,
c) the complete linkage methode!

Draw a dendrogram for each case!

Exercise 41 Fuzzy Clustering

Consider the objective function

$$J(\mathbf{X}, \mathbf{B}, \mathbf{U}) = \sum_{i=1}^{c} \sum_{j=1}^{n} u_{ij}\, d^2(\beta_i, \vec{x}_j),$$

which is to be minimized under the constraint

$$\forall j \in \{1, \ldots, n\} : \quad \sum_{i=1}^{c} \sqrt{u_{ij}} = 1.$$

Derive the update formulae for the membership degrees and the cluster centers using the Euclidean distance. How does the result differ from standard fuzzy clustering with a fuzzifier $w = 2$? (In particular, consider the cluster centers.)

Additional Exercise Expectation Maximization

In exercises 37 and 38 we processed the one-dimensional data set

$$1, 3, 4, 5, 8, 10, 11, 12$$

with fuzzy clustering and expectation maximization. In doing so we discovered a difference in the behavior of the membership degrees. In order to show that this difference does not result from the method, but from the distribution assumption, we apply the expectation maximization algorithm again. However, this time we do not assume a mixture of normal/Gaussian distributions, but a mixture of two Cauchy functions

$$f_{\text{cauchy}}(x; \mu; \sigma^2) = \frac{1}{\pi\sigma} \cdot \frac{1}{\frac{(x-\mu)^2}{\sigma^2} + 1}.$$

As in exercise 38 we assume that the prior probabilities are fixed to $\frac{1}{2}$ and the variances to 1. Likewise, the initial cluster centers are $\mu_1 = 1$ and $\mu_2 = 5$. Compute one expectation step and one maximization step!