

# Fuzzy Systems

## Fuzzy Logic

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# Outline

## 1. Complement

Strict and Strong Negations

Families of Negations

Representation of Negations

## 2. Intersection and Union

## 3. Implication

# Fuzzy Complement/Fuzzy Negation

## Definition

Let  $X$  be a given set and  $\mu \in \mathcal{F}(X)$ . Then the *complement*  $\bar{\mu}$  can be defined pointwise by  $\bar{\mu}(x) := \sim(\mu(x))$  where  $\sim : [0, 1] \rightarrow [0, 1]$  satisfies the conditions

$$\sim(0) = 1, \quad \sim(1) = 0$$

and

for  $x, y \in [0, 1]$ ,  $x \leq y \implies \sim x \geq \sim y$  ( $\sim$  is non-increasing).

Abbreviation:  $\sim x := \sim(x)$

# Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1], x < y \implies \sim x > \sim y$  ( $\sim$  is strictly decreasing)
- $\sim$  is continuous
- $\sim \sim x = x$  for all  $x \in [0, 1]$  ( $\sim$  is involutive)

According to conditions, two subclasses of negations are defined:

## Definition

A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive, too.

$\sim x = 1 - x^2$ , for instance, is strict, not strong, thus not involutive

# Families of Negations

standard negation:

$$\sim x = 1 - x$$

threshold negation:

$$\sim_{\theta}(x) = \begin{cases} 1 & \text{if } x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

Cosine negation:

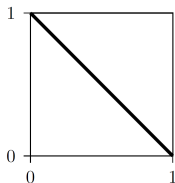
$$\sim x = \frac{1}{2}(1 + \cos(\pi x))$$

Sugeno negation:

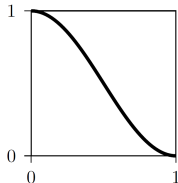
$$\sim_{\lambda}(x) = \frac{1-x}{1+\lambda x}, \quad \lambda > -1$$

Yager negation:

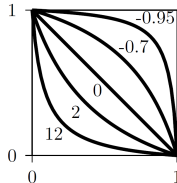
$$\sim_{\lambda}(x) = (1 - x^{\lambda})^{\frac{1}{\lambda}}$$



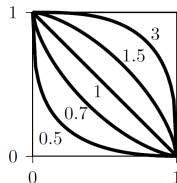
standard



cosine



Sugeno



Yager

## Two Extreme Negations

$$\textit{intuitionistic negation } \sim_i(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\textit{dual intuitionistic negation } \sim_{di}(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Both negations are not strictly increasing, not continuous, not involutive

Thus they are neither strict nor strong

They are “optimal” since their notions are nearest to crisp negation

$\sim_i$  and  $\sim_{di}$  are two extreme cases of negations

For any negation  $\sim$  the following holds

$$\sim_i \leq \sim \leq \sim_{di}$$

## Inverse of a Strict Negation

Any strict negation  $\sim$  is strictly decreasing and continuous.

Hence one can define its inverse  $\sim^{-1}$ .

$\sim^{-1}$  is also strict but in general differs from  $\sim$ .

$\sim^{-1} = \sim$  if and only if  $\sim$  is involutive.

Every strict negation  $\sim$  has a unique value  $0 < s_{\sim} < 1$  such that  $\sim s_{\sim} = s_{\sim}$ .

$s_{\sim}$  is called *membership crossover point*.

$A(a) > s_{\sim}$  if and only if  $A^c(a) < s_{\sim}$  where  $A^c$  is defined via  $\sim$ .

$\sim^{-1}(s_{\sim}) = s_{\sim}$  always holds as well.

# Representation of Negations

**Any strong negation can be obtained from standard negation.**

Let  $a, b \in \mathbb{R}$ ,  $a \leq b$ .

Let  $\varphi : [a, b] \rightarrow [a, b]$  be continuous and strictly increasing.

$\varphi$  is called *automorphism* of the interval  $[a, b] \subset \mathbb{R}$ .

## Theorem

*A function  $\sim : [0, 1] \rightarrow [0, 1]$  is a strong negation if and only if there exists an automorphism  $\varphi$  of the unit interval such that for all  $x \in [0, 1]$  the following holds*

$$\sim_{\varphi}(x) = \varphi^{-1}(1 - \varphi(x)).$$

$\sim_{\varphi}(x) = \varphi^{-1}(1 - \varphi(x))$  is called  $\varphi$ -transform of the standard negation.



# Outline

## 1. Complement

## 2. Intersection and Union

Triangular Norms and Conorms

De Morgan Triplet

Examples

The Special Role of Minimum and Maximum

Continuous Archimedean t-norms and t-conorms

Families of Operations

## 3. Implication

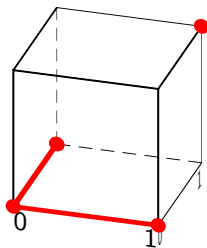
# Classical Intersection and Union

Classical set intersection represents logical conjunction.

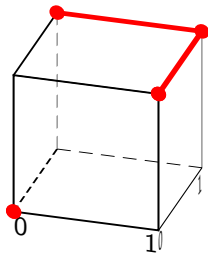
Classical set union represents logical disjunction.

Generalization from  $\{0, 1\}$  to  $[0, 1]$  as follows:

$x \wedge y$	0	1
0	0	0
1	0	1



$x \vee y$	0	1
0	0	1
1	1	1



# Fuzzy Intersection and Union

Let  $A, B$  be fuzzy subsets of  $X$ , i.e.  $A, B \in \mathcal{F}(X)$ .

Their **intersection** and **union** can be defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \quad \text{where} \quad \top : [0, 1]^2 \rightarrow [0, 1]$$

$$(A \cup B)(x) = \perp(A(x), B(x)) \quad \text{where} \quad \perp : [0, 1]^2 \rightarrow [0, 1].$$

# Triangular Norms and Conorms I

$\top$  is a *triangular norm (t-norm)*  $\iff \top$  satisfies conditions T1-T4

$\perp$  is a *triangular conorm (t-conorm)*  $\iff \perp$  satisfies C1-C4

for all  $x, y \in [0, 1]$ , the following laws hold

## Identity Law

$$\mathbf{T1:} \quad \top(x, 1) = x \quad (A \cap X = A)$$

$$\mathbf{C1:} \quad \perp(x, 0) = x \quad (A \cup \emptyset = A).$$

## Commutativity

$$\mathbf{T2:} \quad \top(x, y) = \top(y, x) \quad (A \cap B = B \cap A),$$

$$\mathbf{C2:} \quad \perp(x, y) = \perp(y, x) \quad (A \cup B = B \cup A).$$

# Triangular Norms and Conorms II

for all  $x, y, z \in [0, 1]$ , the following laws hold

## Associativity

$$\mathbf{T3:} \quad \mathbb{T}(x, \mathbb{T}(y, z)) = \mathbb{T}(\mathbb{T}(x, y), z) \quad (A \cap (B \cap C) = (A \cap B) \cap C),$$

$$\mathbf{C3:} \quad \perp(x, \perp(y, z)) = \perp(\perp(x, y), z) \quad (A \cup (B \cup C) = (A \cup B) \cup C).$$

## Monotonicity

$y \leq z$  implies

$$\mathbf{T4:} \quad \mathbb{T}(x, y) \leq \mathbb{T}(x, z)$$

$$\mathbf{C4:} \quad \perp(x, y) \leq \perp(x, z).$$

## Triangular Norms and Conorms III

$\top$  is a *triangular norm (t-norm)*  $\iff \top$  satisfies conditions T1-T4

$\perp$  is a *triangular conorm (t-conorm)*  $\iff \perp$  satisfies C1-C4

Both identity law and monotonicity respectively imply

$$\forall x \in [0, 1] : \top(0, x) = 0,$$

$$\forall x \in [0, 1] : \perp(1, x) = 1,$$

for any *t-norm*  $\top : \top(x, y) \leq \min(x, y)$ ,

for any *t-conorm*  $\perp : \perp(x, y) \geq \max(x, y)$ .

note:  $x = 1 \Rightarrow T(0, 1) = 0$  and

$x \leq 1 \Rightarrow T(x, 0) \leq T(1, 0) = T(0, 1) = 0$

# De Morgan Triplet I

For every  $\top$  and strong negation  $\sim$ , one can define  $t$ -conorm  $\perp$  by

$$\perp(x, y) = \sim \top(\sim x, \sim y), \quad x, y \in [0, 1].$$

Additionally, in this case  $\top(x, y) = \sim \perp(\sim x, \sim y)$ ,  $x, y \in [0, 1]$ .

$\perp, \top$  are called *N-dual t-conorm* and *N-dual t-norm* to  $\top, \perp$ , resp.

In case of the standard negation  $\sim x = 1 - x$  for  $x \in [0, 1]$ ,  
N-dual  $\perp$  and  $\top$  are called *dual t-conorm* and *dual t-norm*, resp.

$\perp(x, y) = \sim \top(\sim x, \sim y)$  expresses “fuzzy ” De Morgan’s law.

note: De Morgan’s laws  $(A \cup B)^c = A^c \cap B^c$ ,  $(A \cap B)^c = A^c \cup B^c$

# De Morgan Triplet II

## Definition

The triplet  $(\top, \perp, \sim)$  is called *De Morgan triplet* if and only if

$\top$  is *t*-norm,  $\perp$  is *t*-conorm,  $\sim$  is strong negation,

$\top, \perp$  and  $\sim$  satisfy  $\perp(x, y) = \sim \top(\sim x, \sim y)$ .

In the following, some important De Morgan triplets will be shown, only the most frequently used and important ones.

In all cases, the standard negation  $\sim x = 1 - x$  is considered.

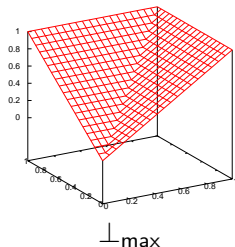
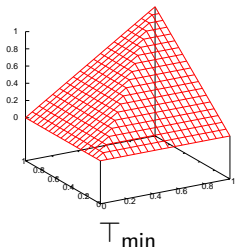


# The Minimum and Maximum I

$$\top_{\min}(x, y) = \min(x, y), \quad \perp_{\max}(x, y) = \max(x, y)$$

Minimum is the greatest  $t$ -norm and max is the weakest  $t$ -conorm.

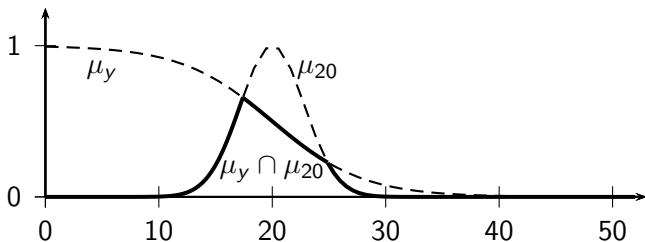
$\top(x, y) \leq \min(x, y)$  and  $\perp(x, y) \geq \max(x, y)$  for any  $\top$  and  $\perp$



## The Minimum and Maximum II

$\top_{\min}$  and  $\perp_{\max}$  can be easily processed numerically and visually, e.g. linguistic values *young* and *approx. 20* described by  $\mu_y$ ,  $\mu_{20}$ .

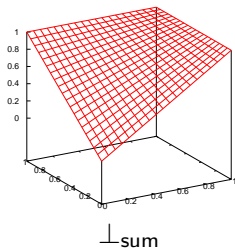
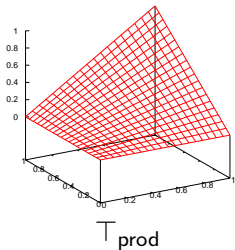
$\top_{\min}(\mu_y, \mu_{20})$  is shown below.



# The Product and Probabilistic Sum

$$\top_{\text{prod}}(x, y) = x \cdot y, \quad \perp_{\text{sum}}(x, y) = x + y - x \cdot y$$

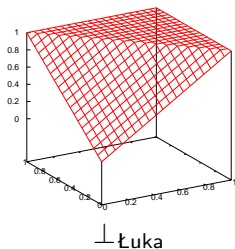
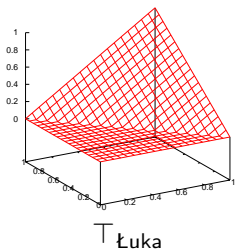
Note that use of product and its dual has nothing to do with probability theory.



# The Łukasiewicz $t$ -norm and $t$ -conorm

$$\top_{\text{Łuka}}(x, y) = \max\{0, x + y - 1\}, \quad \perp_{\text{Łuka}}(x, y) = \min\{1, x + y\}$$

$\top_{\text{Łuka}}, \perp_{\text{Łuka}}$  are also called *bold intersection* and *bounded sum*.

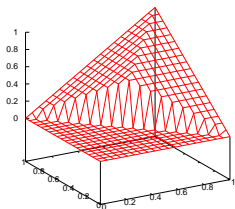


# The Nilpotent Minimum and Maximum

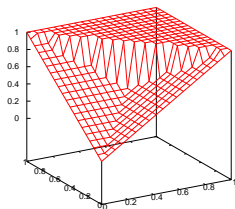
$$\top_{\min_0}(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\perp_{\max_1}(x, y) = \begin{cases} \max(x, y) & \text{if } x + y < 1 \\ 1 & \text{otherwise} \end{cases}$$

New since found in 1992 and independently rediscovered in 1995.



$\top_{\min_0}$



$\perp_{\max_1}$

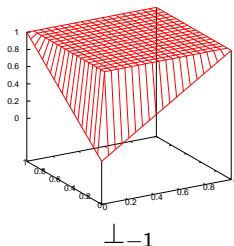
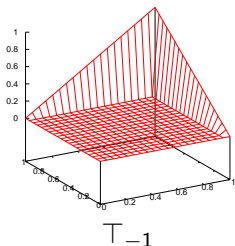
## The Drastic Product and Sum

$$\top_{-1}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

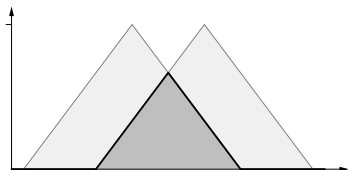
$$\perp_{-1}(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$\top_{-1}$  is the weakest  $t$ -norm,  $\perp_{-1}$  is the strongest  $t$ -conorm.

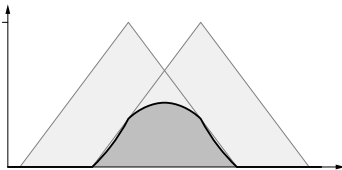
$\top_{-1} \leq \top \leq \top_{\min}$ ,  $\perp_{\max} \leq \perp \leq \perp_{-1}$  for any  $\top$  and  $\perp$



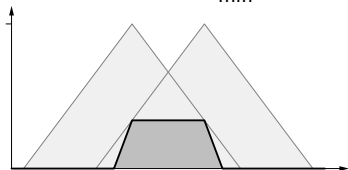
## Examples of Fuzzy Intersections



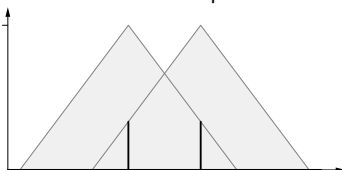
$t$ -norm  $\top_{\min}$



$t$ -norm  $\top_{\text{prod}}$



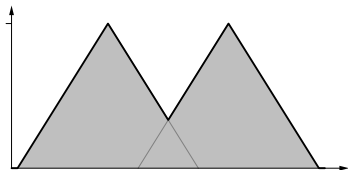
$t$ -norm  $\top_{\text{Łuka}}$



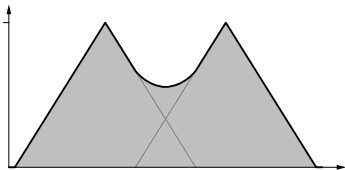
$t$ -norm  $\top_{-1}$

Note that all fuzzy intersections are contained within upper left graph and lower right one.

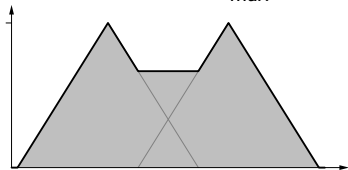
## Examples of Fuzzy Unions



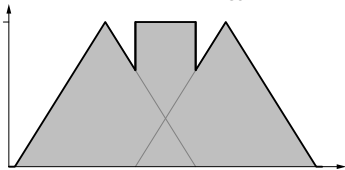
$t$ -conorm  $\perp_{\max}$



$t$ -conorm  $\perp_{\text{sum}}$



$t$ -conorm  $\perp_{\text{Łuka}}$



$t$ -conorm  $\perp_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.



# The Special Role of Minimum and Maximum I

$\top_{\min}$  and  $\perp_{\max}$  play key role for intersection and union, resp.

In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all  $x, y, z \in [0, 1]$ :

## Distributivity

$$\perp_{\max}(x, \top_{\min}(y, z)) = \top_{\min}(\perp_{\max}(x, y), \perp_{\max}(x, z)),$$

$$\top_{\min}(x, \perp_{\max}(y, z)) = \perp_{\max}(\top_{\min}(x, y), \top_{\min}(x, z))$$

## Continuity

$\top_{\min}$  and  $\perp_{\max}$  are continuous.

# The Special Role of Minimum and Maximum II

## Strict monotonicity on the diagonal

$x < y$  implies  $\top_{\min}(x, x) < \top_{\min}(y, y)$  and  $\perp_{\max}(x, x) < \perp_{\max}(y, y)$ .

## Idempotency

$$\top_{\min}(x, x) = x, \quad \perp_{\max}(x, x) = x$$

## Absorption

$$\top_{\min}(x, \perp_{\max}(x, y)) = x, \quad \perp_{\max}(x, \top_{\min}(x, y)) = x$$

## Non-compensation

$x < y < y$  imply  $\top_{\min}(x, z) \neq \top_{\min}(y, y)$  and  
 $\perp_{\max}(x, z) \neq \perp_{\max}(y, y)$ .

## The Special Role of Minimum and Maximum III

Is  $(\mathcal{F}(X), \top_{\min}, \perp_{\max}, \sim)$  a boolean algebra?

Consider the properties (B1)-(B9) of any Boolean algebra.

For  $(\mathcal{F}(X), \top_{\min}, \perp_{\max}, \sim)$  with strong negation  $\sim$  only complementary (B7) does not hold.

Hence  $(\mathcal{F}(X), \top_{\min}, \perp_{\max}, \sim)$  is a *completely distributive lattice* with identity element  $\mu_X$  and zero element  $\mu_{\emptyset}$ .

**No lattice**  $(\mathcal{F}(X), \top, \perp, \sim)$  forms a Boolean algebra

due to the fact that complementary (B7) does not hold:

- There is no complement/negation  $\sim$  with  $\top(A, \sim A) = \mu_{\emptyset}$ .
- There is no complement/negation  $\sim$  with  $\perp(A, \sim A) = \mu_X$ .

# Complementary Property of Fuzzy Sets I

Using fuzzy sets, it's **impossible** to keep up a Boolean algebra.

Verify, e.g. that law of contradiction is violated, *i.e.*

$$(\exists x \in X)(A \cap A^c)(x) \neq \emptyset.$$

We use min, max and strong negation  $\sim$  as fuzzy set operators.

So we need to show that

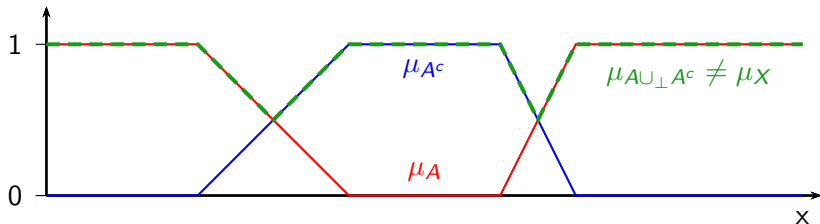
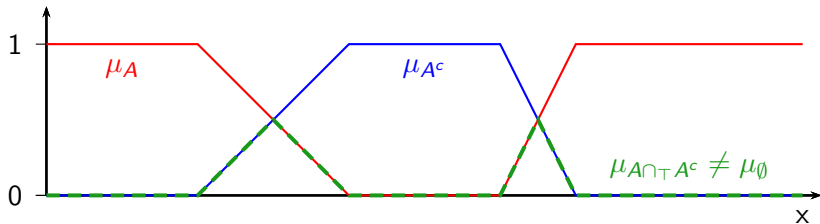
$$\min\{A(x), 1 - A(x)\} = 0$$

is violated for at least one  $x \in X$ .

easy: This Equation is violated for all  $A(x) \in (0, 1)$ .

It is satisfied only for  $A(x) \in \{0, 1\}$ .

# Complementary Property of Fuzzy Sets II: Example



# The concept of a pseudoinverse

## Definition

Let  $f : [a, b] \rightarrow [c, d]$  be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to  $f$  is the function  $f^{(-1)} : [c, d] \rightarrow [a, b]$  defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$

## Continuous Archimedean $t$ -norms and $t$ -conorms

broad class of problems relates to representation of multi-place functions by composition of “simpler” functions, e.g.

$$K(x, y) = g(f(x) + f(y))$$

So, one should consider suitable subclass of all  $t$ -norms.

### Definition

A  $t$ -norm  $\top$  is

- (a) *continuous* if  $\top$  as function is continuous on unit interval,
- (b) *Archimedean* if  $\top$  is continuous and  $\top(x, x) < x$  for all  $x \in ]0, 1[$ .

### Definition

A  $t$ -conorm  $\perp$  is

- (a) *continuous* if  $\perp$  as function is continuous on unit interval,
- (b) *Archimedean* if  $\perp$  is continuous and  $\perp(x, x) > x$  for all  $x \in ]0, 1[$ .

# Continuous Archimedean $t$ -norms

## Theorem

A  $t$ -norm  $\top$  is continuous and Archimedean if and only if there exists a strictly decreasing and continuous function  $f : [0, 1] \rightarrow [0, \infty]$  with  $f(1) = 0$  such that

$$\top(x, y) = f^{(-1)}(f(x) + f(y)) \quad (1)$$

where

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \leq f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of  $f$ . Moreover, this representation is unique up to a positive multiplicative constant.

$\top$  is generated by  $f$  if  $\top$  has representation (1).

$f$  is called *additive generator* of  $\top$ .



## Additive Generators of $t$ -norms – Examples

Find an additive generator  $f$  of  $\top_{\text{Łuka}} = \max\{x + y - 1, 0\}$ .

for instance  $f_{\text{Łuka}}(x) = 1 - x$

then,  $f_{\text{Łuka}}^{(-1)}(x) = \max\{1 - x, 0\}$

thus  $\top_{\text{Łuka}}(x, y) = f_{\text{Łuka}}^{(-1)}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator  $f$  of  $\top_{\text{prod}} = x \cdot y$ .

to be discussed in the exercise

hint: use of logarithmic and exponential function

# Continuous Archimedean $t$ -conorms

## Theorem

A  $t$ -conorm  $\perp$  is continuous and Archimedean if and only if there exists a strictly increasing and continuous function  $g : [0, 1] \rightarrow [0, \infty]$  with  $g(0) = 0$  such that

$$\perp(x, y) = g^{(-1)}(g(x) + g(y)) \quad (2)$$

where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of  $g$ . Moreover, this representation is unique up to a positive multiplicative constant.

$\perp$  is generated by  $g$  if  $\perp$  has representation (2).

$g$  is called *additive generator* of  $\perp$ .

## Additive Generators of $t$ -conorms – Two Examples

Find an additive generator  $g$  of  $\perp_{\text{Luka}} = \min\{x + y, 1\}$ .

for instance  $g_{\text{Luka}}(x) = x$

then,  $g_{\text{Luka}}^{(-1)}(x) = \min\{x, 1\}$

thus  $\perp_{\text{Luka}}(x, y) = g_{\text{Luka}}^{(-1)}(g_{\text{Luka}}(x) + g_{\text{Luka}}(y))$

Find an additive generator  $g$  of  $\perp_{\text{sum}} = x + y - x \cdot y$ .

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

# Hamacher Family I

$$\top_{\alpha}(x, y) = \frac{x \cdot y}{\alpha + (1 - \alpha)(x + y + x \cdot y)}, \quad \alpha \geq 0,$$

$$\perp_{\beta}(x, y) = \frac{x + y + (\beta - 1) \cdot x \cdot y}{1 + \beta \cdot x \cdot y}, \quad \beta \geq -1,$$

$$\sim_{\gamma}(x) = \frac{1 - x}{1 + \gamma x}, \quad \gamma > -1$$

## Theorem

$(\top, \perp, \sim)$  is a De Morgan triplet such that

$$\top(x, y) = \top(x, z) \implies y = z,$$

$$\perp(x, y) = \perp(x, z) \implies y = z,$$

$$\forall z \leq x \exists y, y' \text{ such that } \top(x, y) = z, \perp(z, y') = x$$

and  $\top$  and  $\perp$  are rational functions if and only if there are numbers  $\alpha \geq 0$ ,  $\beta \geq -1$  and  $\gamma > -1$  such that  $\alpha = \frac{1+\beta}{1+\gamma}$  and  $\top = \top_{\alpha}$ ,  $\perp = \perp_{\beta}$  and  $\sim = \sim_{\gamma}$ .

## Hamacher Family II

Additive generators  $f_\alpha$  of  $\top_\alpha$  are

$$f_\alpha = \begin{cases} \frac{1-x}{x} & \text{if } \alpha = 0 \\ \log \frac{\alpha + (1-\alpha)x}{x} & \text{if } \alpha > 0. \end{cases}$$

Each member of these families is strict  $t$ -norm and strict  $t$ -conorm, respectively.

Members of this family of  $t$ -norms are decreasing functions of parameter  $\alpha$ .

# Sugeno-Weber Family I

For  $\lambda > 1$  and  $x, y \in [0, 1]$ , define

$$\top_{\lambda}(x, y) = \max \left\{ \frac{x + y - 1 + \lambda xy}{1 + \lambda}, 0 \right\},$$

$$\perp_{\lambda}(x, y) = \min \{x + y + \lambda xy, 1\}.$$

$\lambda = 0$  leads to  $\top_{\text{Łuka}}$  and  $\perp_{\text{Łuka}}$ , resp.

$\lambda \rightarrow \infty$  results in  $\top_{\text{prod}}$  and  $\perp_{\text{sum}}$ , resp.

$\lambda \rightarrow -1$  creates  $\top_{-1}$  and  $\perp_{-1}$ , resp.

## Sugeno-Weber Family II

Additive generators  $f_\lambda$  of  $\top_\lambda$  are

$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$  are increasing functions of parameter  $\lambda$ .

Additive generators of  $\perp_\lambda$  are  $g_\lambda(x) = 1 - f_\lambda(x)$ .

## Yager Family

For  $0 < p < \infty$  and  $x, y \in [0, 1]$ , define

$$\begin{aligned} \top_p(x, y) &= \max \left\{ 1 - ((1-x)^p + (1-y)^p)^{1/p}, 0 \right\}, \\ \perp_p(x, y) &= \min \left\{ (x^p + y^p)^{1/p}, 1 \right\}. \end{aligned}$$

Additive generators of  $\top_p$  are

$$f_p(x) = (1-x)^p,$$

and of  $\perp_p$  are

$$g_p(x) = x^p.$$

$\{\top_p\}_{0 < p < \infty}$  are strictly increasing in  $p$ .

Note that  $\lim_{p \rightarrow +\infty} \top_p = \top_{\text{Łuka}}$ .



# Outline

## 1. Complement

## 2. Intersection and Union

## 3. Implication

S-Implications

R-Implications

QL-Implications

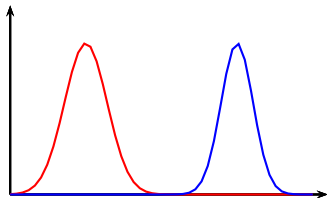
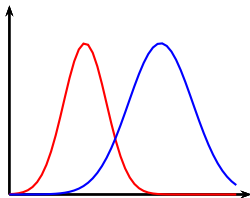
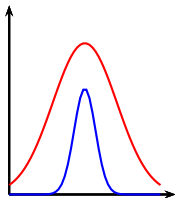
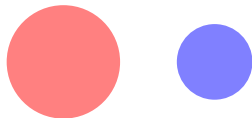
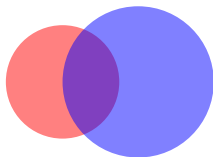
Axioms

List of Fuzzy Implications

Selection of Fuzzy Implications

# Fuzzy Implications

crisp:  $x \in A \Rightarrow x \in B$ , fuzzy:  $x \in \mu \Rightarrow x \in \mu'$



## Definitions of Fuzzy Implications

One way of defining  $I$  is to use  $\forall a, b \in \{0, 1\}$

$$I(a, b) = \neg a \vee b.$$

In fuzzy logic, disjunction and negation are  $t$ -conorm and fuzzy complement, resp., thus  $\forall a, b \in [0, 1]$

$$I(a, b) = \perp(\sim a, b).$$

Another way in classical logic is  $\forall a, b \in \{0, 1\}$

$$I(a, b) = \max \{x \in \{0, 1\} \mid a \wedge x \leq b\}.$$

In fuzzy logic, conjunction represents  $t$ -norm, thus  $\forall a, b \in [0, 1]$

$$I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}.$$

So, classical definitions are equal, fuzzy extensions are not.

## Definitions of Fuzzy Implications

$I(a, b) = \perp(\sim a, b)$  may also be written as either

$$I(a, b) = \neg a \vee (a \wedge b) \quad \text{or}$$

$$I(a, b) = (\neg a \wedge \neg b) \vee b.$$

Fuzzy logical extensions are thus, respectively,

$$I(a, b) = \perp(\sim a, \top(a, b)),$$

$$I(a, b) = \perp(\top(\sim a, \sim b), b)$$

where  $(\top, \perp, \sim)$  must be a *De Morgan triplet*.

So again, classical definitions are equal, fuzzy extensions are not.

reason: Law of absorption of negation does not hold in fuzzy logic.

## S-Implications

Implications based on  $I(a, b) = \perp(\sim a, b)$  are called **S-implications**.

Symbol  $S$  is often used to denote  $t$ -conorms.

Four well-known  $S$ -implications are based on  $\sim a = 1 - a$ :

Name	$I(a, b)$	$\perp(a, b)$
Kleene-Dienes	$I_{\max}(a, b) = \max(1 - a, b)$	$\max(a, b)$
Reichenbach	$I_{\text{sum}}(a, b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{\perp}(a, b) = \min(1, 1 - a + b)$	$\min(1, a + b)$
largest	$I_{-1}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$

# S-Implications

The drastic sum  $\perp_{-1}$  leads to the largest S-implication  $I_{-1}$  due to the following theorem:

## Theorem

*Let  $\perp_1, \perp_2$  be t-conorms such that  $\perp_1(a, b) \leq \perp_2(a, b)$  for all  $a, b \in [0, 1]$ . Let  $I_1, I_2$  be S-implications based on same fuzzy complement  $\sim$  and  $\perp_1, \perp_2$ , respectively. Then  $I_1(a, b) \leq I_2(a, b)$  for all  $a, b \in [0, 1]$ .*

Since  $\perp_{-1}$  leads to the largest S-implication, similarly,  $\perp_{\max}$  leads to the smallest S-implication  $I_{\max}$ .

Furthermore,

$$I_{\max} \leq I_{\text{sum}} \leq I_{\perp} \leq I_{-1}.$$

## $R$ -Implications

$I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}$  leads to  $R$ -**implications**.

Symbol  $R$  represents close connection to residuated semigroup.

Three well-known  $R$ -implications are based on  $\sim a = 1 - a$ :

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a, b) = \sup \{x \mid \min(a, x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a, b) = \sup \{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz  $t$ -norm leads to **Łukasiewicz implication**

$$I_{\text{Ł}}(a, b) = \sup \{x \mid \max(0, a + x - 1) \leq b\} = \min(1, 1 - a + b).$$

# R-Implications

Name	Formula	$\top(a, b) =$
Gödel	$I_{\min}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$	$\min(a, b)$
Goguen	$I_{\text{prod}}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$	$ab$
Łukasiewicz	$I_{\text{Ł}}(a, b) = \min(1, 1 - a + b)$	$\max(0, a + b - 1)$
largest	$I_{\text{L}}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1, & \text{otherwise} \end{cases}$	not defined

$I_{\text{L}}$  is actually the limit of all  $R$ -implications.

It serves as least upper bound.

It cannot be defined by  $I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}$ .



# R-Implications

## Theorem

Let  $\top_1, \top_2$  be  $t$ -norms such that  $\top_1(a, b) \leq \top_2(a, b)$  for all  $a, b \in [0, 1]$ . Let  $I_1, I_2$  be  $R$ -implications based on  $\top_1, \top_2$ , respectively. Then  $I_1(a, b) \geq I_2(a, b)$  for all  $a, b \in [0, 1]$ .

It follows that Gödel  $I_{\min}$  is the smallest  $R$ -implication.

Furthermore,

$$I_{\min} \leq I_{\text{prod}} \leq I_{\text{L}} \leq I_{\text{L}}.$$

## QL-Implications

Implications based on  $I(a, b) = \perp(\sim a, \top(a, b))$  are called **QL-implications** (QL from quantum logic).

Four well-known QL-implications are based on  $\sim a = 1 - a$ :

- Standard min and max lead to **Zadeh implication**

$$I_Z(a, b) = \max[1 - a, \min(a, b)].$$

- The algebraic product and sum lead to

$$I_p(a, b) = 1 - a + a^2 b.$$

- Using  $\top_{\perp}$  and  $\perp_{\perp}$  leads to **Kleene-Dienes implication** again.
- Using  $\top_{-1}$  and  $\perp_{-1}$  leads to

$$I_q(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

# Axioms

All  $I$  come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following axioms:

- $a \leq b$  implies  $I(a, x) \geq I(b, x)$  (*monotonicity in 1st argument*)
- $a \leq b$  implies  $I(x, a) \leq I(x, b)$  (*monotonicity in 2nd argument*)
- $I(0, a) = 1$  (*dominance of falsity*)
- $I(1, b) = b$  (*neutrality of truth*)
- $I(a, a) = 1$  (*identity*)
- $I(a, I(b, c)) = I(b, I(a, c))$  (*exchange property*)
- $I(a, b) = 1$  if and only if  $a \leq b$  (*boundary condition*)
- $I(a, b) = I(\sim b, \sim a)$  for fuzzy complement  $\sim$  (*contraposition*)
- $I$  is a continuous function (*continuity*)

# Generator Function

$I$  that satisfy all listed axioms are characterized by this theorem:

## Theorem

A function  $I : [0, 1]^2 \rightarrow [0, 1]$  satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement  $\sim$  if and only if there exists a strict increasing continuous function  $f : [0, 1] \rightarrow [0, \infty)$  such that  $f(0) = 0$ ,

$$I(a, b) = f^{(-1)}(f(1) - f(a) + f(b))$$

for all  $a, b \in [0, 1]$ , and

$$\sim a = f^{-1}(f(1) - f(a))$$

for all  $a \in [0, 1]$ .

## Example

Consider  $f_\lambda(a) = \ln(1 + \lambda a)$  with  $a \in [0, 1]$  and  $\lambda > 0$ .

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy complement generated by  $f$  for all  $a \in [0, 1]$  is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all  $a, b \in [0, 1]$  is thus

$$I_\lambda(a, b) = \min \left( 1, \frac{1 - a + b + \lambda b}{1 + \lambda a} \right).$$

If  $\lambda \in (-1, 0)$ , then  $I_\lambda$  is called **pseudo-Lukasiewicz implication**.

# List of Fuzzy Implications

Name	Class	Form $I(a, b) =$	Axioms	Complement
Gaines-Rescher		$\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$	1-8	$1 - a$
Gödel	R	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$	1-7	
Goguen	R	$\begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases}$	1-7, 9	
Kleene-Dienes	S, QL	$\max(1 - a, b)$	1-4, 6, 8, 9	$1 - a$
Łukasiewicz	R, S	$\min(1, 1 - a + b)$	1-9	$1 - a$
Pseudo-Łukasiewicz 1	R, S	$\min \left[ 1, \frac{1-a+(1+\lambda)b}{1+\lambda a} \right]$	1-9	$\frac{1-a}{1+\lambda a}, (\lambda > -1)$
Pseudo-Łukasiewicz 2	R, S	$\min [1, 1 - a^w + b^w]$	1-9	$(1 - a^w)^{\frac{1}{w}}, (w > 0)$
Reichenbach	S	$1 - a + ab$	1-4, 6, 8, 9	$1 - a$
Wu		$\begin{cases} 1 & \text{if } a \leq b \\ \min(1 - a, b) & \text{otherwise} \end{cases}$	1-3,5,7,8	$1 - a$
Zadeh	QL	$\max[1 - a, \min(a, b)]$	1-4, 9	$1 - a$

# Which Fuzzy Implication?

Since the meaning of  $I$  is not unique, we must resolve the following question:

Which  $I$  should be used for calculating the fuzzy relation  $R$ ?

Hence meaningful criteria are needed.

They emerge from various fuzzy inference rules, *i.e.* modus ponens, modus tollens, hypothetical syllogism.