OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Systems

Fuzzy Logic

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Outline

1. **Complement**
   - Strict and Strong Negations
   - Families of Negations
   - Representation of Negations
2. **Intersection and Union**
3. **Implication**

Fuzzy Complement/Fuzzy Negation

Definition

Let X be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) :=\sim (\mu(x))$ where $\sim : [0,1] \to [0,1]$ satisfies the conditions

$$\sim (0) = 1, \quad \sim (1) = 0$$

and

$$\text{for } x, y \in [0,1],\ x \leq y \implies \sim x \geq \sim y \quad (\sim \text{ is non-increasing}).$$

Abbreviation: $\sim x :=\sim (x)$

Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1],\ x < y \Longrightarrow \sim x > \sim y$ ($\sim$ is strictly decreasing)
- $\sim$ is continuous
- $\sim\sim x = x$ for all $x \in [0, 1]$ ($\sim$ is involutive)

According to conditions, two subclasses of negations are defined:

Definition

A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive, too.

$\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

Families of Negations

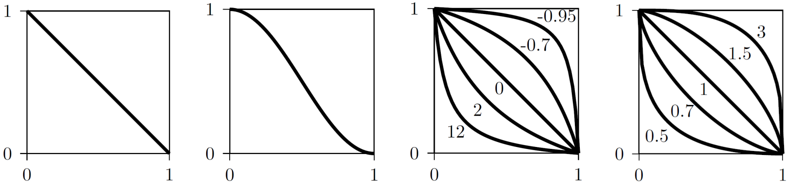
standard negation: $\sim x = 1 - x$

threshold negation: $\sim_\theta (x) = \begin{cases} 1 & \text{if } x \le \theta \\ 0 & \text{otherwise} \end{cases}$

Cosine negation: $\sim x = \frac{1}{2}(1 + \cos(\pi x))$

Sugeno negation: $\sim_\lambda (x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1$

Yager negation: $\sim_\lambda (x) = (1 - x^\lambda)^{\frac{1}{\lambda}}$



standard cosine Sugeno Yager

Two Extreme Negations

intuitionistic negation $\sim_i(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

dual intuitionistic negation $\sim_{di}(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$

Both negations are not strictly increasing, not continuous, not involutive

Thus they are neither strict nor strong

They are “optimal” since their notions are nearest to crisp negation

$\sim_i$ and $\sim_{di}$ are two extreme cases of negations

For any negation $\sim$ the following holds

$$\sim_i \leq \sim \leq \sim_{di}$$

Inverse of a Strict Negation

Any strict negation $\sim$ is strictly decreasing and continuous.

Hence one can define its inverse $\sim^{-1}$.

$\sim^{-1}$ is also strict but in general differs from $\sim$.

$\sim^{-1}=\sim$ if and only if $\sim$ is involutive.

Every strict negation $\sim$ has a unique value $0 < s_\sim < 1$ such that $\sim s_\sim = s_\sim$.

$s_\sim$ is called *membership crossover point*.

$A(a) > s_\sim$ if and only if $A^c(a) < s_\sim$ where A^c is defined via $\sim$.

$\sim^{-1}(s_\sim) = s_\sim$ always holds as well.

Representation of Negations

Any strong negation can be obtained from standard negation.

Let $a, b \in \mathbb{R},\ a \leq b$.

Let $\varphi : [a, b] \to [a, b]$ be continuous and strictly increasing.

φ is called *automorphism* of the interval $[a, b] \subset \mathbb{R}$.

Theorem

A function $\sim : [0, 1] \to [0, 1]$ *is a strong negation if and only if there exists an automorphism* φ *of the unit interval such that for all* $x \in [0, 1]$ *the following holds*

$$\sim_\varphi (x) = \varphi^{-1}(1 - \varphi(x)).$$

$\sim_\varphi (x) = \varphi^{-1}(1 - \varphi(x))$ is called φ-transform of the standard negation.

Outline

Classical Intersection and Union

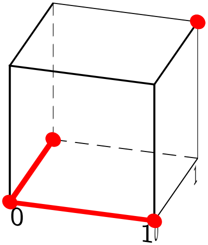
Classical set intersection represents logical conjunction.

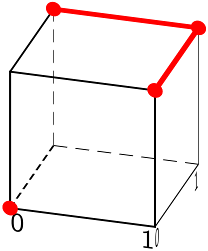
Classical set union represents logical disjunction.

Generalization from $\{0, 1\}$ to $[0, 1]$ as follows:

$x \wedge y$	0	1
0	0	0
1	0	1

$x \vee y$	0	1
0	0	1
1	1	1





Fuzzy Intersection and Union

Let A, B be fuzzy subsets of X, *i.e.* $A, B \in \mathcal{F}(X)$.

Their **intersection** and **union** can be defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \quad \text{where} \quad \top : [0,1]^2 \to [0,1]$$
$$(A \cup B)(x) = \bot(A(x), B(x)) \quad \text{where} \quad \bot : [0,1]^2 \to [0,1].$$

Triangular Norms and Conorms I

$\top$ is a *triangular norm (t-norm)* $\Longleftrightarrow$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\Longleftrightarrow$ $\bot$ satisfies C1-C4

for all $x, y \in [0, 1]$, the following laws hold

Identity Law

T1: $\top(x, 1) = x \quad (A \cap X = A)$

C1: $\bot(x, 0) = x \quad (A \cup \emptyset = A)$.

Commutativity

T2: $\top(x, y) = \top(y, x) \quad (A \cap B = B \cap A)$,

C2: $\bot(x, y) = \bot(y, x) \quad (A \cup B = B \cup A)$.

Triangular Norms and Conorms II

for all $x, y, z \in [0, 1]$, the following laws hold

Associativity

T3: $\top(x, \top(y, z)) = \top(\top(x, y), z) \quad (A \cap (B \cap C)) = ((A \cap B) \cap C)$,
C3: $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z) \quad (A \cup (B \cup C)) = ((A \cup B) \cup C)$.

Monotonicity

$y \leq z$ implies
T4: $\top(x, y) \leq \top(x, z)$
C4: $\bot(x, y) \leq \bot(x, z)$.

Triangular Norms and Conorms III

$\top$ is a *triangular norm (t-norm)* $\iff$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\iff$ $\bot$ satisfies C1-C4

Both identity law and monotonicity respectively imply

$$\begin{aligned} \forall x \in [0,1] &: \top(0,x) = 0, \\ \forall x \in [0,1] &: \bot(1,x) = 1, \\ \text{for any } t\text{-norm } \top &: \top(x,y) \leq \min(x,y), \\ \text{for any } t\text{-conorm } \bot &: \bot(x,y) \geq \max(x,y). \end{aligned}$$

note: $x = 1 \Rightarrow T(0,1) = 0$ and
$x \leq 1 \Rightarrow T(x,0) \leq T(1,0) = T(0,1) = 0$

De Morgan Triplet I

For every $\top$ and strong negation $\sim$, one can define *t*-conorm $\bot$ by

$$\bot(x,y) = \sim \top(\sim x, \sim y), \quad x,y \in [0,1].$$

Additionally, in this case $\top(x,y) = \sim \bot(\sim x, \sim y), \ \ x,y \in [0,1]$.

$\bot, \top$ are called *N-dual t-conorm* and *N-dual t-norm* to $\top, \bot$, resp.

In case of the standard negation $\sim x = 1 - x$ for $x \in [0,1]$,
N-dual $\bot$ and $\top$ are called *dual t-conorm* and *dual t-norm*, resp.

$\bot(x,y) = \sim \top(\sim x, \sim y)$ expresses “fuzzy” De Morgan’s law.

note: De Morgan’s laws $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

De Morgan Triplet II

Definition

The triplet $(\top, \bot, \sim)$ is called *De Morgan triplet* if and only if

$\top$ is t-norm, $\bot$ is t-conorm, $\sim$ is strong negation,

$\top, \bot$ and $\sim$ satisfy $\bot(x, y) =\sim \top(\sim x, \sim y)$.

In the following, some important De Morgan triplets will be shown,

only the most frequently used and important ones.

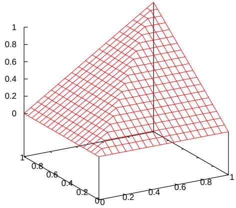
In all cases, the standard negation $\sim x = 1 - x$ is considered.

The Minimum and Maximum I

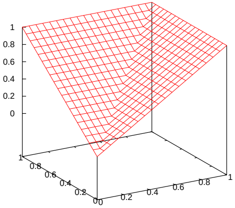
$\top_{\text{min}}(x, y) = \min(x, y), \quad \bot_{\text{max}}(x, y) = \max(x, y)$

Minimum is the greatest *t*-norm and max is the weakest *t*-conorm.

$\top(x, y) \leq \min(x, y)$ and $\bot(x, y) \geq \max(x, y)$ for any $\top$ and $\bot$



$\top_{\text{min}}$



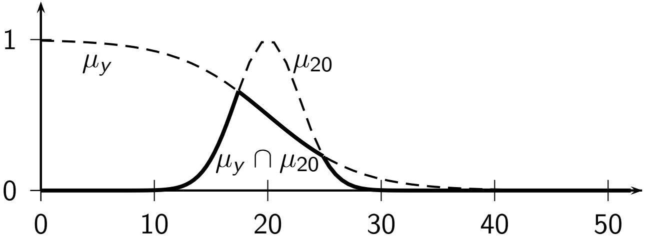
$\bot_{\text{max}}$

The Minimum and Maximum II

$\top_{\min}$ and $\bot_{\max}$ can be easily processed numerically and visually,

e.g. linguistic values *young* and *approx. 20* described by μ_y, μ_{20}.

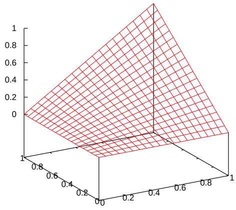
$\top_{\min}(\mu_y, \mu_{20})$ is shown below.



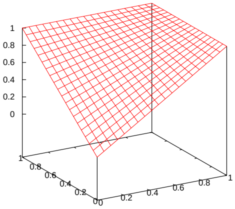
The Product and Probabilistic Sum

$$\top_{\text{prod}}(x, y) = x \cdot y, \quad \bot_{\text{sum}}(x, y) = x + y - x \cdot y$$

Note that use of product and its dual has nothing to do with probability theory.



$\top_{\text{prod}}$

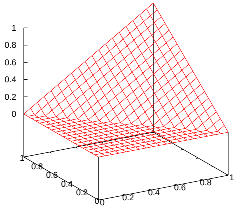


$\bot_{\text{sum}}$

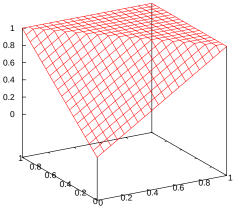
The Łukasiewicz *t*-norm and *t*-conorm

$\top_{\text{Łuka}}(x, y) = \max\{0,\ x + y - 1\}, \quad \bot_{\text{Łuka}}(x, y) = \min\{1,\ x + y\}$

$\top_{\text{Łuka}}$, $\bot_{\text{Łuka}}$ are also called *bold intersection* and *bounded sum*.



$\top_{\text{Łuka}}$



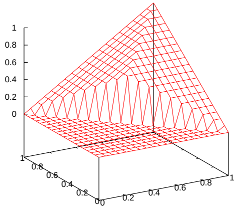
$\bot_{\text{Łuka}}$

The Nilpotent Minimum and Maximum

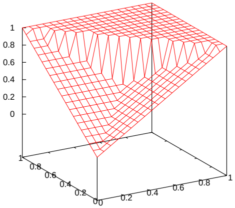
$$\top_{\min_0}(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\bot_{\max_1}(x, y) = \begin{cases} \max(x, y) & \text{if } x + y < 1 \\ 1 & \text{otherwise} \end{cases}$$

Found in 1992 and independently rediscovered in 1995.



$\top_{\min_0}$



$\bot_{\max_1}$

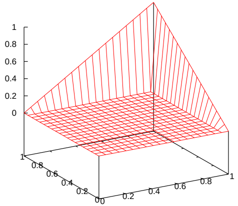
The Drastic Product and Sum

$$\top_{-1}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

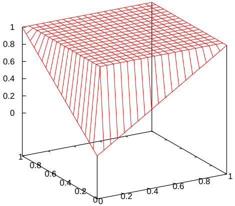
$$\bot_{-1}(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$\top_{-1}$ is the weakest t-norm, $\bot_{-1}$ is the strongest t-conorm.

$\top_{-1} \le \top \le \top_{\min}, \quad \bot_{\max} \le \bot \le \bot_{-1}$ for any $\top$ and $\bot$

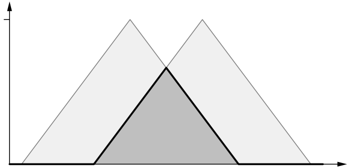


$\top_{-1}$

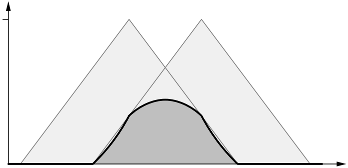


$\bot_{-1}$

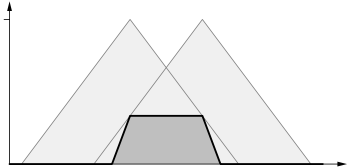
Examples of Fuzzy Intersections



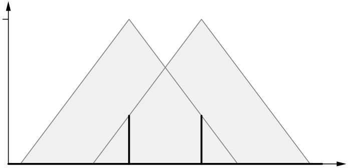
t-norm $\top_{\min}$



t-norm $\top_{\text{prod}}$



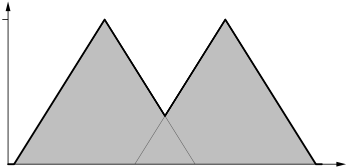
t-norm $\top_{\text{Łuka}}$



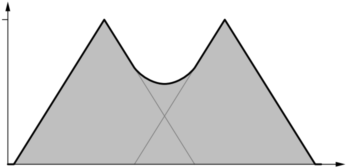
t-norm $\top_{-1}$

Note that all fuzzy intersections are contained within upper left graph and lower right one.

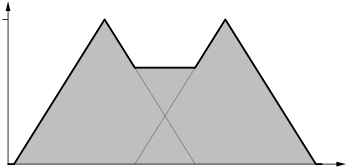
Examples of Fuzzy Unions



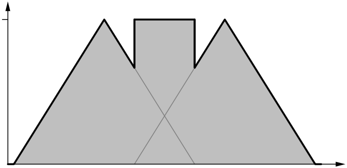
t-conorm $\bot_{\text{max}}$



t-conorm $\bot_{\text{sum}}$



t-conorm $\bot_{\text{Łuka}}$



t-conorm $\bot_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.

The Special Role of Minimum and Maximum I

$\top_{\min}$ and $\bot_{\max}$ play key role for intersection and union, resp.

In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0, 1]$:

Distributivity

$$\bot_{\max}(x, \top_{\min}(y, z)) = \top_{\min}(\bot_{\max}(x, y), \bot_{\max}(x, z)),$$
$$\top_{\min}(x, \bot_{\max}(y, z)) = \bot_{\max}(\top_{\min}(x, y), \top_{\min}(x, z))$$

Continuity

$\top_{\min}$ and $\bot_{\max}$ are continuous.

The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

$x < y$ implies $\top_{\min}(x, x) < \top_{\min}(y, y)$ and $\bot_{\max}(x, x) < \bot_{\max}(y, y)$.

Idempotency

$$\top_{\min}(x, x) = x, \quad \bot_{\max}(x, x) = x$$

Absorption

$$\top_{\min}(x, \bot_{\max}(x, y)) = x, \quad \bot_{\max}(x, \top_{\min}(x, y)) = x$$

Non-compensation

$x < y < z$ imply $\top_{\min}(x, z) \neq \top_{\min}(y, y)$ and
$\bot_{\max}(x, z) \neq \bot_{\max}(y, y)$.

The Special Role of Minimum and Maximum III

Is $(\mathcal{F}(X), \top_{\min}, \bot_{\max}, \sim)$ a boolean algebra?

Consider the properties (B1)-(B9) of any Boolean algebra.

For $(\mathcal{F}(X), \top_{\min}, \bot_{\max}, \sim)$ with strong negation $\sim$ only complementary (B7) does not hold.

Hence $(\mathcal{F}(X), \top_{\min}, \bot_{\max}, \sim)$ is a *completely distributive lattice* with identity element μ_X and zero element $\mu_\emptyset$.

No lattice $(\mathcal{F}(X), \top, \bot, \sim)$ forms a Boolean algebra

due to the fact that complementary (B7) does not hold:

- There is no complement/negation $\sim$ with $\top(A, \sim A) = \mu_\emptyset$.
- There is no complement/negation $\sim$ with $\bot(A, \sim A) = \mu_X$.

Complementary Property of Fuzzy Sets I

Using fuzzy sets, it's **impossible** to keep up a Boolean algebra.

Verify, *e.g.* that law of contradiction is violated, *i.e.*

$$(\exists x \in X)(A \cap A^c)(x) \neq \emptyset.$$

We use $\min, \max$ and strong negation $\sim$ as fuzzy set operators.

So we need to show that

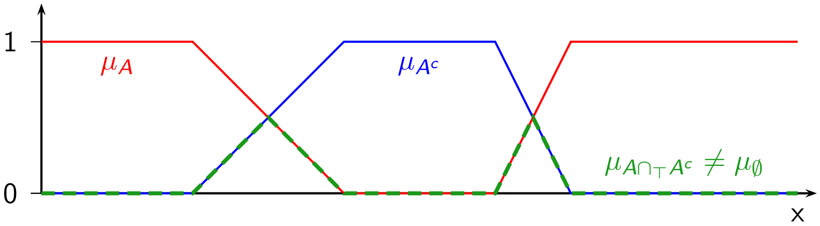
$$\min\{A(x), 1 - A(x)\} = 0$$

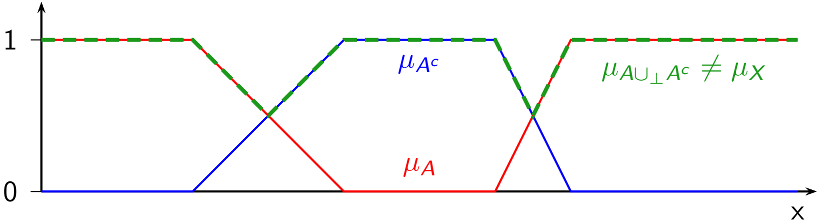
is violated for at least one $x \in X$.

easy: This Equation is violated for all $A(x) \in (0, 1)$.

It is satisfied only for $A(x) \in \{0, 1\}$.

Complementary Property of Fuzzy Sets II: Example





The concept of a pseudoinverse

Definition

Let $f : [a, b] \to [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to f is the function $f^{(-1)} : [c, d] \to [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$

Continuous Archimedean t-norms and t-conorms

broad class of problems relates to representation of multi-place functions by composition of a "simpler" function, *e.g.*

$$K(x, y) = f^{(-1)}(f(x) + f(y))$$

So, one should consider suitable subclass of all t-norms.

Definition

A t-norm $\top$ is

(a) *continuous* if $\top$ as function is continuous on unit interval,

(b) *Archimedean* if $\top$ is continuous and $\top(x, x) < x$ for all $x \in]0, 1[$.

Definition

A t-conorm $\perp$ is

(a) *continuous* if $\perp$ as function is continuous on unit interval,

(b) *Archimedean* if $\perp$ is continuous and $\perp(x, x) > x$ for all $x \in]0, 1[$.

Continuous Archimedean t-norms

Theorem

A t-norm $\top$ is continuous and Archimedean if and only if there exists a strictly decreasing and continuous function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ such that

$$\top(x,y) = f^{(-1)}(f(x) + f(y)) \qquad (1)$$

where

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \leq f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of f. Moreover, this representation is unique up to a positive multiplicative constant.

$\top$ *is generated by f* if $\top$ has representation (1).

f is called *additive generator* of $\top$.

Additive Generators of t-norms – Examples

Find an additive generator f of $\top_{\text{Łuka}}(x, y) = \max\{x + y - 1,\ 0\}$.

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f_{\text{Łuka}}^{(-1)}(x) = \max\{1 - x,\ 0\}$

thus $\top_{\text{Łuka}}(x, y) = f_{\text{Łuka}}^{(-1)}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator f of $\top_{\text{prod}}(x, y) = x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Continuous Archimedean t-conorms

Theorem

A t-conorm $\bot$ is continuous and Archimedean if and only if there exists a strictly increasing and continuous function $g : [0, 1] \to [0, \infty]$ with $g(0) = 0$ such that

$$\bot(x, y) = g^{(-1)}(g(x) + g(y)) \qquad (2)$$

where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of g. Moreover, this representation is unique up to a positive multiplicative constant.

$\bot$ *is generated by g* if $\bot$ has representation (2).

g is called *additive generator* of $\bot$.

Additive Generators of t-conorms – Two Examples

Find an additive generator g of $\bot_{\textbf{Łuka}}(x, y) = \min\{x + y,\ 1\}$.

for instance $g_{\text{Łuka}}(x) = x$

then, $g_{\text{Łuka}}^{(-1)}(x) = \min\{x,\ 1\}$

thus $\bot_{\text{Łuka}}(x, y) = g_{\text{Łuka}}^{(-1)}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y))$

Find an additive generator g of $\bot_{\textbf{sum}}(x, y) = x + y - x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

Hamacher Family I

$$\top_\alpha(x,y) = \frac{x \cdot y}{\alpha + (1-\alpha)(x+y+x\cdot y)}, \quad \alpha \geq 0,$$
$$\bot_\beta(x,y) = \frac{x+y+(\beta-1)\cdot x\cdot y}{1+\beta\cdot x\cdot y}, \quad \beta \geq -1,$$
$$\sim_\gamma(x) = \frac{1-x}{1+\gamma x}, \quad \gamma > -1$$

Theorem

$(\top, \bot, \sim)$ *is a De Morgan triplet such that*

$$\top(x,y) = \top(x,z) \Longrightarrow y = z,$$
$$\bot(x,y) = \bot(x,z) \Longrightarrow y = z,$$
$$\forall z \leq x \ \exists y, y' \text{ such that } \top(x,y) = z, \ \bot(z,y') = x$$

and $\top$ *and* $\bot$ *are rational functions if and only if there are numbers* $\alpha \geq 0, \ \beta \geq -1$ *and* $\gamma > -1$ *such that* $\alpha = \frac{1+\beta}{1+\gamma}$ *and* $\top = \top_\alpha, \ \bot = \bot_\beta$ *and* $\sim = \sim_\gamma$.

Hamacher Family II

Additive generators f_α of $\top_\alpha$ are

$$f_\alpha = \begin{cases} \frac{1-x}{x} & \text{if } \alpha = 0 \\ \log \frac{\alpha+(1-\alpha)x}{x} & \text{if } \alpha > 0. \end{cases}$$

Each member of these families is strict t-norm and strict t-conorm, respectively.

Members of this family of t-norms are decreasing functions of parameter α.

Sugeno-Weber Family I

For $\lambda > 1$ and $x, y \in [0, 1]$, define

$$\top_\lambda(x, y) = \max\left\{\frac{x + y - 1 + \lambda xy}{1 + \lambda},\ 0\right\},$$

$$\bot_\lambda(x, y) = \min\{x + y + \lambda xy,\ 1\}.$$

$\lambda = 0$ leads to $\top_{\text{Łuka}}$ and $\bot_{\text{Łuka}}$, resp.

$\lambda \to \infty$ results in $\top_{\text{prod}}$ and $\bot_{\text{sum}}$, resp.

$\lambda \to -1$ creates $\top_{-1}$ and $\bot_{-1}$, resp.

Sugeno-Weber Family II

Additive generators f_λ of $\top_\lambda$ are

$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter λ.

Additive generators of $\bot_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$.

Yager Family

For $0 < p < \infty$ and $x, y \in [0, 1]$, define

$$\top_p(x, y) = \max\left\{1 - ((1-x)^p + (1-y)^p)^{1/p},\ 0\right\},$$

$$\bot_p(x, y) = \min\left\{(x^p + y^p)^{1/p},\ 1\right\}.$$

Additive generators of $\top_p$ are

$$f_p(x) = (1-x)^p,$$

and of $\bot_p$ are

$$g_p(x) = x^p.$$

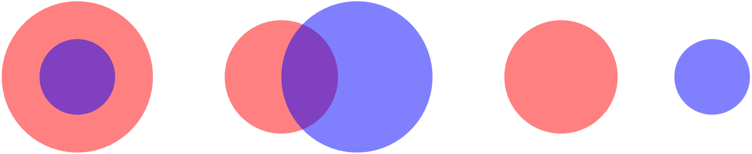
$\{\top_p\}_{0<p<\infty}$ are strictly increasing in p.

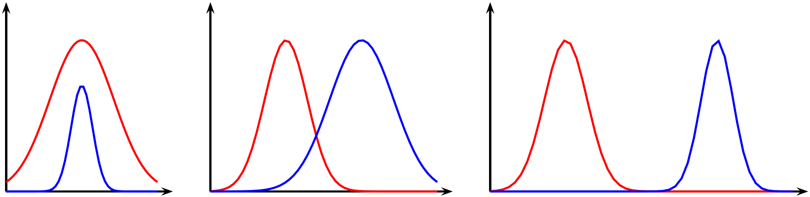
Note that $\lim_{p\to+0} \top_p = \top_{\text{Łuka}}$.

Outline

Fuzzy Implications

crisp: $x \in A \Rightarrow x \in B$, **fuzzy:** $x \in \mu \Rightarrow x \in \mu'$





Definitions of Fuzzy Implications

One way of defining I is to use $\forall a, b \in \{0, 1\}$

$$I(a, b) = \neg a \vee b.$$

In fuzzy logic, disjunction and negation are *t*-conorm and fuzzy complement, resp., thus $\forall a, b \in [0, 1]$

$$I(a, b) = \bot(\sim a, b).$$

Another way in classical logic is $\forall a, b \in \{0, 1\}$

$$I(a, b) = \max \{x \in \{0, 1\} \mid a \wedge x \leq b\}.$$

In fuzzy logic, conjunction represents *t*-norm, thus $\forall a, b \in [0, 1]$

$$I(a, b) = \sup \{x \in [0, 1] \mid \top(a, x) \leq b\}.$$

So, classical definitions are equal, fuzzy extensions are not.

Definitions of Fuzzy Implications

$I(a, b) = \bot(\sim a, b)$ may also be written as either

$$I(a, b) = \neg a \vee (a \wedge b) \qquad \text{or}$$
$$I(a, b) = (\neg a \wedge \neg b) \vee b.$$

Fuzzy logical extensions are thus, respectively,

$$I(a, b) = \bot(\sim a, \top(a, b)),$$
$$I(a, b) = \bot(\top(\sim a, \sim b), b)$$

where $(\top, \bot, n)$ must be a *De Morgan triplet*.

So again, classical definitions are equal, fuzzy extensions are not.

reason: Law of absorption of negation does not hold in fuzzy logic.

S-Implications

Implications based on $I(a, b) = \bot(\sim a, b)$ are called S-**implications**.

Symbol S is often used to denote t-conorms.

Four well-known S-implications are based on $\sim a = 1 - a$:

Name	$I(a, b)$	$\bot(a, b)$
Kleene-Dienes	$I_{\max}(a, b) = \max(1 - a, b)$	$\max(a, b)$
Reichenbach	$I_{\text{sum}}(a, b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{Ł}(a, b) = \min(1,\ 1 - a + b)$	$\min(1,\ a + b)$
largest	$I_{-1}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$

S-Implications

The drastic sum $\bot_{-1}$ leads to the largest S-implication I_{-1} due to the following theorem:

Theorem
Let $\bot_1, \bot_2$ be t-conorms such that $\bot_1(a,b) \le \bot_2(a,b)$ for all $a, b \in [0,1]$. Let I_1, I_2 be S-implications based on same fuzzy complement $\sim$ and $\bot_1, \bot_2$, respectively. Then $I_1(a,b) \le I_2(a,b)$ for all $a, b \in [0,1]$.

Since $\bot_{-1}$ leads to the largest S-implication, similarly, $\bot_{\max}$ leads to the smallest S-implication $I_{\max}$.

Furthermore,

$$I_{\max} \le I_{\text{sum}} \le I_{Ł} \le I_{-1}.$$

R-Implications

$I(a, b) = \sup\{x \in [0, 1] \mid \top(a, x) \leq b\}$ leads to R-**implications**.

Symbol R represents close connection to residuated semigroup.

Three well-known R-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a, b) = \sup\{x \mid \min(a, x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a, b) = \sup\{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz t-norm leads to **Łukasiewicz implication**

$$I_{Ł}(a, b) = \sup\{x \mid \max(0, a + x - 1) \leq b\} = \min(1,\ 1 - a + b).$$

R-Implications

Name	Formula	$\top(a,b) =$
Gödel	$I_{\min}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$	$\min(a,b)$
Goguen	$I_{\text{prod}}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$	ab
Łukasiewicz	$I_{Ł}(a,b) = \min(1,\ 1 - a + b)$	$\max(0,\ a + b - 1)$
largest	$I_{L}(a,b) = \begin{cases} b, & \text{if } a = 1 \\ 1, & \text{otherwise} \end{cases}$	not defined

I_L is actually the limit of all R-implications.

It serves as least upper bound.

It cannot be defined by $I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}$.

R-Implications

Theorem
Let $\top_1, \top_2$ be t-norms such that $\top_1(a, b) \leq \top_2(a, b)$ for all $a, b \in [0, 1]$. Let I_1, I_2 be R-implications based on $\top_1, \top_2$, respectively. Then $I_1(a, b) \geq I_2(a, b)$ for all $a, b \in [0, 1]$.

It follows that Gödel $I_{\min}$ is the smallest R-implication.

Furthermore,

$$I_{\min} \leq I_{\text{prod}} \leq I_{\text{Ł}} \leq I_{\text{L}}.$$

QL-Implications

Implications based on $I(a, b) = \bot(\sim a, \top(a, b))$ are called QL-**implications** (QL from quantum logic).

Four well-known QL-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**

$$I_Z(a, b) = \max[1 - a, \min(a, b)].$$

- The algebraic product and sum lead to

$$I_\mathrm{p}(a, b) = 1 - a + a^2 b.$$

- Using $\top_Ł$ and $\bot_Ł$ leads to **Kleene-Dienes implication** again.
- Using $\top_{-1}$ and $\bot_{-1}$ leads to

$$I_\mathrm{q}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

Axioms

All I come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following axioms:

1) $a \leq b$ implies $I(a, x) \geq I(b, x)$ *(monotonicity in 1st argument)*
2) $a \leq b$ implies $I(x, a) \leq I(x, b)$ *(monotonicity in 2nd argument)*
3) $I(0, a) = 1$ *(dominance of falsity)*
4) $I(1, b) = b$ *(neutrality of truth)*
5) $I(a, a) = 1$ *(identity)*
6) $I(a, I(b, c)) = I(b, I(a, c))$ *(exchange property)*
7) $I(a, b) = 1$ if and only if $a \leq b$ *(boundary condition)*
8) $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$ *(contraposition)*
9) I is a continuous function *(continuity)*

Generator Function

I that satisfy all listed axioms are characterized by this theorem:

Theorem

A function $I : [0,1]^2 \to [0,1]$ satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement $\sim$ if and only if there exists a strict increasing continuous function $f : [0,1] \to [0,\infty)$ such that $f(0) = 0$,

$$I(a,b) = f^{(-1)}(f(1) - f(a) + f(b))$$

for all $a, b \in [0,1]$, and

$$\sim a = f^{-1}(f(1) - f(a))$$

for all $a \in [0,1]$.

Example

Consider $f_\lambda(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$.

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy complement generated by f for all $a \in [0, 1]$ is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_\lambda(a, b) = \min\left(1, \frac{1 - a + b + \lambda b}{1 + \lambda a}\right).$$

If $\lambda \in (-1, 0)$, then I_λ is called **pseudo-Łukasiewicz implication**.

List of Fuzzy Implications

Name	Class	Form $I(a, b) =$	Axioms	Complement
Gaines-Rescher		$\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$	1–8	$1 - a$
Gödel	R	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$	1–7	
Goguen	R	$\begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases}$	1–7, 9	
Kleene-Dienes	S,QL	$\max(1 - a, b)$	1–4, 6, 8, 9	$1 - a$
Łukasiewicz	R, S	$\min(1, 1 - a + b)$	1–9	$1 - a$
Pseudo-Łukasiewicz 1	R, S	$\min\left[1, \frac{1-a+(1+\lambda)b}{1+\lambda a}\right]$	1–9	$\frac{1-a}{1+\lambda a}$, $(\lambda > -1)$
Pseudo-Łukasiewicz 2	R, S	$\min\left[1, 1 - a^w + b^w\right]$	1–9	$(1 - a^w)^{\frac{1}{w}}$, $(w > 0)$
Reichenbach	S	$1 - a + ab$	1–4, 6, 8, 9	$1 - a$
Wu		$\begin{cases} 1 & \text{if } a \leq b \\ \min(1 - a, b) & \text{otherwise} \end{cases}$	1–3,5,7,8	$1 - a$
Zadeh	QL	$\max[1 - a, \min(a, b)]$	1–4, 9	$1 - a$

Which Fuzzy Implication?

Since the meaning of I is not unique, we must resolve the following question:

Which I should be used for calculating the fuzzy relation R?

Hence meaningful criteria are needed.

They emerge from various fuzzy inference rules, *i.e.* modus ponens, modus tollens, hypothetical syllogism.