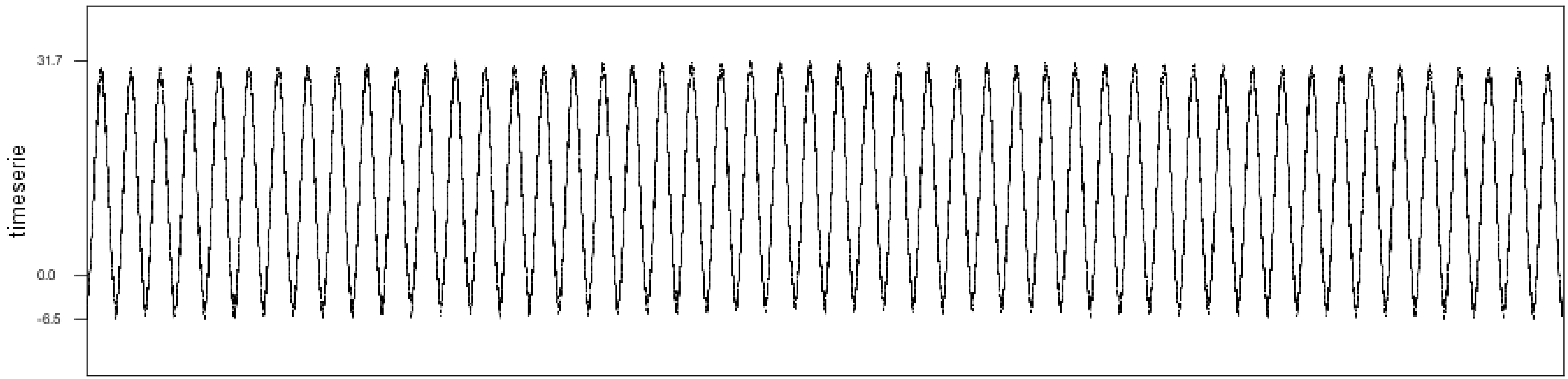
Time Series Analysis

Time Series

- **Motivation**
- **Decomposition Models**
  - Additive models, multiplicative models
- **Global Approaches**
  - Regression
  - With and without seasonal component
- **Local Approaches**
  - Moving Averages Smoothing
  - With and without seasonal component
- **Summary**

Motivation: Temperatures

Example: Temperatures data set (fictive)



- The plot shows the average temperature per day for 50 years.
- Is there any trend visible?
- How to extract seasonal effects?

Decomposition Models

- The time series is given as a sequence of values

$$y_1, \ldots, y_t, \ldots, y_n$$

- We assume that every y_t is a composition of (some of) the following components:
  - g_t trend component
  - s_t seasonal component
  - c_t cyclical variation
  - ϵ_t irregular component (random factors, noise)
- Assume a functional dependency:

$$y_t = f(g_t, s_t, c_t, \epsilon_t)$$

Components of Time Series

Trend Component

- Reflects long-term developments.
- Often assumed to be a monotone function of time.
- Represents the actual component we are interested in.

Cyclic Component

- Reflects mid-term developments.
- Models economical cycles such as booms and recessions.
- Variable cycle length.
- We do not consider this component here.

Remark: Often, both components are combined.

Components of Time Series

Seasonal Component

- Reflects short-term developments.
- Constant cycle length (i. e., 12 months)
- Represents changes that (re)occur rather regularly.

Irregular Component

- Represents everything else that cannot be related to the other components.
- Combines irregular changes, random noise and local fluctuations.
- We assume that the values are small and have an average of zero.

Decompositions

Additive Decomposition

$$y_t = g_t + s_t + \epsilon_t$$

- Pure trend model: $y_t = g_t + \epsilon_t$ (stock market, no season)
- Possible extension: $y_t = g_t + s_t + x_t\beta + \epsilon_t$ (calendar effects)

Multiplicative Decomposition

$$y_t = g_t \cdot s_t \cdot \epsilon_t$$

- Seasonal changes may increase with trend.
- Transform into additive model:

$$\tilde{y}_t = \log y_t + \log s_t + \log \epsilon_t$$

Goal: Estimate the components from a given time series, i. e.

$$\hat{g}_t + \hat{s}_t + \epsilon_t \;\approx\; y_t$$

Application: With the estimates, we can compute the

- trend-adjusted series: $y_t - \hat{g}_t$
- season-adjusted series: $y_t - \hat{s}_t$
- We only consider additive models here.

$\Rightarrow$ Additional assumptions necessary in order to find ways to infer the desired components.

Overview

- **Global approach:** There is a fix functional dependence throughout the entire time range. ($\Rightarrow$ regression models)
- **Local approach:** We do not postulate a global model and rather use local estimations to describe the respective components.
- **Seasonal effects:** We have to decide beforehand whether to assume a seasonal component or not.

	Global	Local
without Season	Regression	Smoothing Averages
with Season	Dummy Variables	Smoothing Averages

Global Approach (without Season)

Model: $y_t = g_t + \epsilon_t$

Assumptions:

- No seasonal component: $s_t = 0$
- Depending on g_t, use regression analysis to estimate the parameter(s) to define the trend component.
  - linear trend: $g_t = \beta_0 + \beta_1 t$
  - quadratic trend: $g_t = \beta_0 + \beta_1 t + \beta_2 t^2$
  - polynomial trend: $g_t = \beta_0 + \beta_1 t + \cdots + \beta_q t^q$
  - exponential trend: $g_t = \beta_0 \exp(\beta_1 t)$
  - logistic trend: $g_t = \frac{\beta_0}{\beta_1 + \exp(-\beta_2 t)}$

Global Approach (with Season)

Model: $y_t = s_t + \epsilon_t$ (no trend)

Assumptions:

- No trend component: $g_t = 0$
- Seasonal component does not change from period to period.
- Introduce *dummy variables* for every time span (here: months) that serve as indicator functions to determine to which month a specific t belongs:

$$s_m(t) = \begin{cases} 1, & \text{if } t \text{ belongs to month } m \\ 0, & \text{otherwise} \end{cases}$$

- The seasonal component is then set up as $s_t = \sum_{m=1}^{12} \beta_m s_m(t)$.
- Determine the *monthly effects* β_m with normal least squares method.

Global Approach (with Season)

Model: $y_t = g_t + s_t + \epsilon_t$

Assumptions:

- Estimate $\hat{g}_t$ while temporarily ignoring s_t.
- Estimate s_t from the trend-adjusted $\tilde{y}_t = y_t - \hat{g}_t$.

Model: $y_t = \alpha_1 t + \cdots + \alpha_q t^q + \cdots + \beta_1 s_1(t) + \cdots + \beta_{12} s_{12}(t) + \epsilon_t$

Assumptions:

- Seasonal component does not change from period to period.
- Model the seasonal effects with trigonometric functions:

$$s_t = \beta_0 + \sum_{m=1}^{6} \beta_m \cos\Big(2\pi \frac{m}{12} t\Big) + \sum_{m=1}^{5} \gamma_m \sin\Big(2\pi \frac{m}{12} t\Big)$$

- Determine $\alpha_1, \ldots, \alpha_q, \beta_0, \ldots, \beta_6$ and $\gamma_1, \ldots, \gamma_5$ with normal least squares method.

Local Approach (without Season)

General Idea: Smooth the time series.

- Estimate the trend component g_t at time t as the average of the values around time t.

For a given time series $y_1, \ldots, y_n$, the **Smoothing Average** $y_t^\star$ of order r is defined as follows:

$$y_t^\star = \begin{cases} \dfrac{1}{2k+1} \cdot \displaystyle\sum_{j=-k}^{k} y_{t+j}, & \text{if } r = 2k+1 \\[2em] \dfrac{1}{2k} \cdot \left(\dfrac{1}{2} y_{t-k} + \displaystyle\sum_{j=-k+1}^{k-1} y_{t+j} + \dfrac{1}{2} y_{t+k}\right), & \text{if } r = 2k \end{cases}$$

Local Approach (without Season)

Model: $y_t = g_t + \epsilon_t$

Assumptions:

- In every time frame of width $2k + 1$ the time series can be assumed to be linear.
- ϵ_t averages to zero.
- Then we use the smoothing average to estimate the trend component:

$$\hat{g}_t = y_t^\star$$

Local Approach (with Season)

Model: $y_t = g_t + s_t + \epsilon_t$

Assumptions:

- Seasonal component has period length p (repeats after p points):

$$s_t = s_{t+p}, \quad t = 1, \ldots, n-p$$

- Sum of seasonal values is zero: $\sum_{j=1}^{p} s_j = 0$

- Trend component is linear in time frames of width p (if p is odd) or $p+1$ (if p is even).

- Irregular component averages to zero.

Local Approach (with Season)

Let $k = \frac{p-1}{2}$ (for odd p) or $k = \frac{p}{2}$ (for even p).

Then:

- Estimate the trend component with smoothing average:

$$\hat{g}_t = y_t^\star, \qquad k+1 \le t \le n-k$$

- Estimate the seasonal components $s_1, \ldots, s_p$ as follows:

$$\hat{s}_i = \tilde{s}_i - \frac{1}{p}\sum_{j=1}^{p} \tilde{s}_j \quad \text{with} \quad \tilde{s}_t \frac{1}{m_i - l_i + 1}\sum_{j=l}^{m_i}(y_{i+jp} - y_{i+jp}^\star), \qquad 1 \le i \le p$$

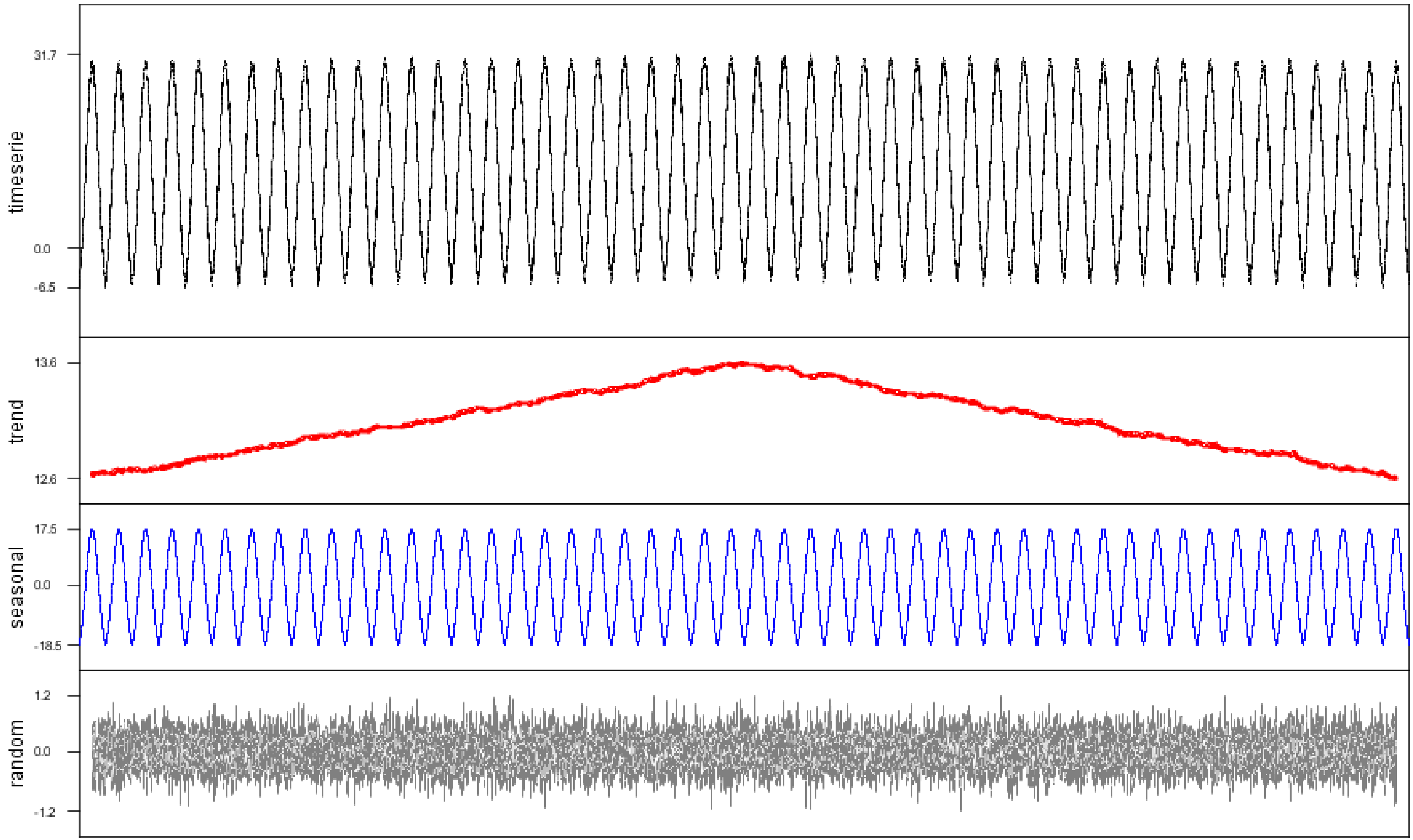
where

$$m_i = \max\left\{m \in \mathbb{N}_0 \mid i + mp \le n - k\right\}$$

and

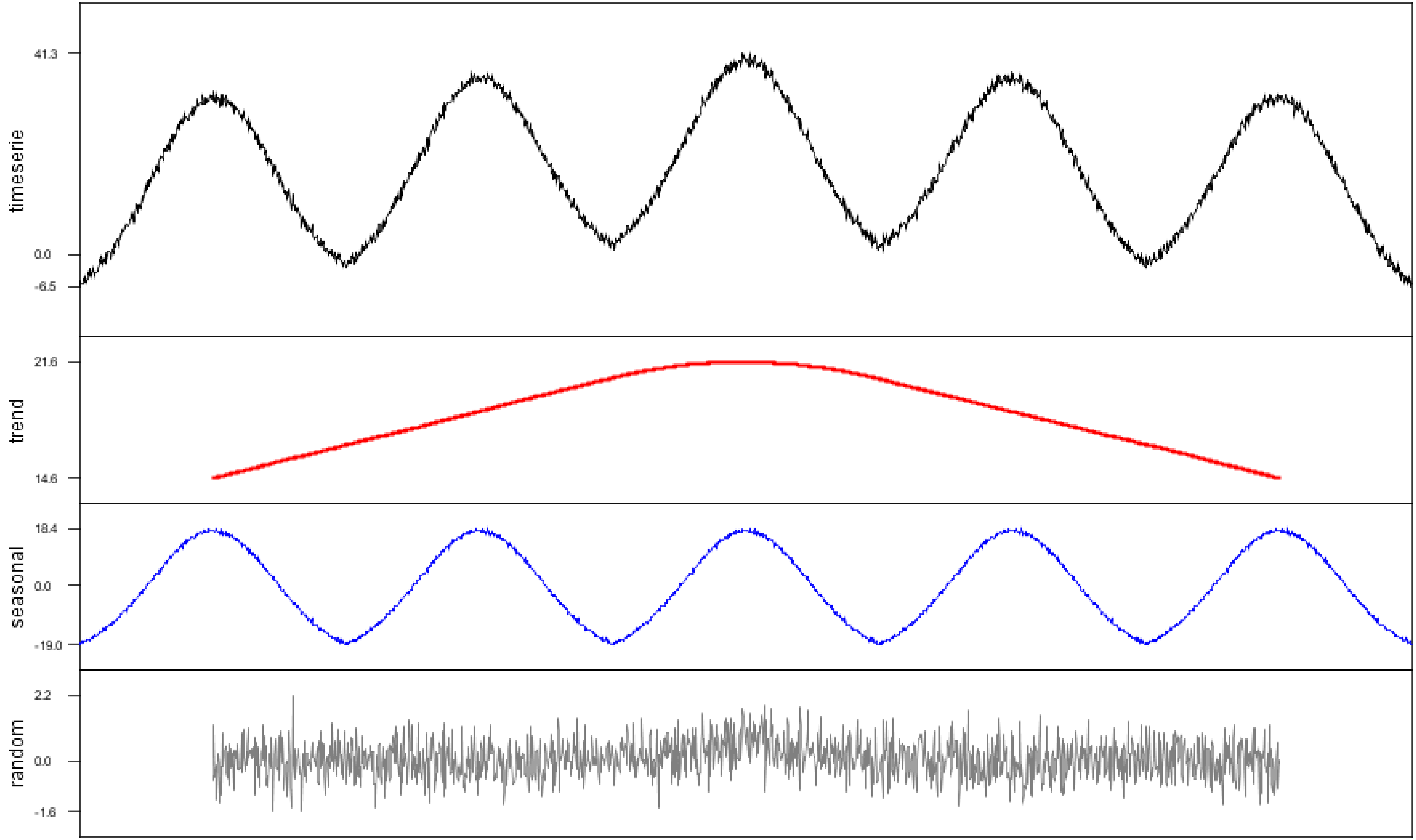
$$l_i = \min\left\{l \in \mathbb{N}_0 \mid i + lp \ge k + 1\right\}$$

Example (from motivation)



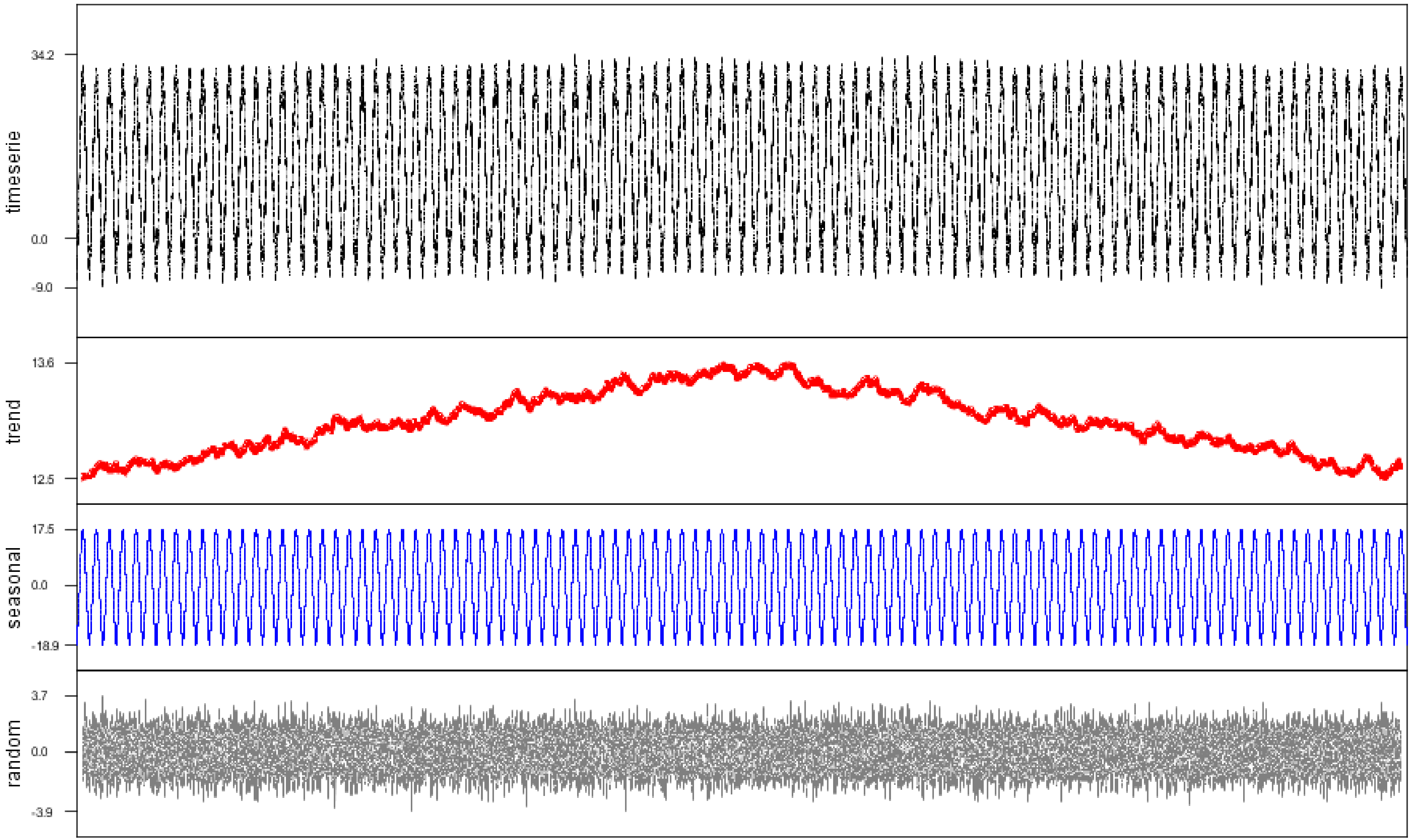
- We can extract an increase and decrease of 1 degree during 50 years even though the amount of noise is more than twice as large than the actual trend.

Example



5 years period, trend ±8 degrees, noise amount ±2 degrees

Example



100 years period, trend ±1 degree, noise amount ±3 degrees

Summary

- **Definition of the problem domain**
  - Consider a time series to be composed of subcomponents.
  - Additive and multiplicative models.
- **Global and local approaches**
  - With and without seasonal components.
- **Robust to noise**
  - Noise can be higher than the trend component itself.